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The structure of multivariate AR and ARMA systems: Regular and singular systems; the single and the mixed frequency case

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ABSTRACT

This paper is concerned with the structure of multivariate AR and ARMA systems. The emphasis is on two "non-standard" cases: We deal with the structure of singular AR and ARMA systems which generate singular spectral densities and with identifiability of ARMA systems from mixed frequency data. In the mixed frequency case we show that, for the case where the MA order is smaller than or equal to the AR order, identifiability can be achieved generically. Furthermore, we demonstrate that for a pure MA system identifiability cannot be achieved. The paper generalizes the results obtained in Anderson et al. (2015) for the AR case.

1. Introduction

In many areas of application, several single time series are available and information exceeding the univariate information in every single time series is of interest. The main reasons for joint modeling of multivariate time series are:

1. The analysis of dynamic relations between time series.
2. Improvement of forecasts and nowcasts.
3. Extraction of factors common to all time series.

Here we restrict ourselves to multivariate AR and ARMA systems and to the stationary case. Whereas the standard (regular) AR case is well understood nowadays (see e.g. Lütkepohl (2005)) and this is also more or less true for the standard ARMA case (Reinsel (1993); Lütkepohl (2005); Hannan and Deistler (2012)), there are certain important aspects, in particular for applications in economics and finance, which still need further clarification. Here we consider two of these aspects:

1. Singular AR and ARMA systems, i.e. systems where the output spectrum is singular. Such systems occur in dynamic stochastic general equilibrium (DSGE) models, when the number of shocks is strictly smaller than the number of outputs (see e.g. Komunjer and Ng (2011a,b)), in generalized dynamic factor models, as models for the latent variables or the static factors (see e.g. Forni et al. (2009)), when the dimension of the dynamic factors is strictly smaller than the dimension of the static factors, or simply in the presence of definition equations. Alternative forms of singularity have been discussed in Solo (2008).
2. Modeling a "high frequency" system from mixed frequency data. This problem occurs, for instance, in economic applications where, e.g., financial data are observed more frequently than data related to real economy. The AR case has been treated in e.g. Anderson et al. (2015) and here we extend our results to the ARMA case.

The emphasis of this paper is on structure theory related to the two issues listed above. Here we use the term structure theory for the analysis of the relation between model (i.e. system and noise) parameters of classes of AR and ARMA systems and the corresponding (population) second moments of the observations. A particular part of structure theory is concerned with identifiability. Structure theory is important for subsequent estimation.

We consider multivariate ARMA systems of the form

$$y_t + A_1 y_{t-1} + \cdots + A_p y_{t-p} = B_0 \nu_t + B_1 \nu_{t-1} + \cdots + B_s \nu_{t-s}, \quad t \in \mathbb{Z} \tag{1.1}$$

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where p and s are specified integers, $A_i, B_i \in \mathbb{R}^{n\times n}$ and $(\nu_t | t \in \mathbb{Z})$ is white noise with $\Sigma = \mathbb{E}\left(\nu_t \nu_t^T\right)$. Let

$$a(z) = I_n + A_1 z + \cdots + A_p z^p$$
$$b(z) = B_0 + B_1 z + \cdots + B_s z^s,$$

where z is used for a complex variable as well as for the backward shift on $\mathbb{Z}$.

Throughout we impose the stability condition

$$\det a(z) \neq 0, \quad |z| \leq 1 \tag{1.2}$$

and the miniphase condition

$$\det b(z) \neq 0, \quad |z| < 1. \tag{1.3}$$

The steady state solution of (1.1) is stationary and given by

$$y_t = a^{-1}(z)b(z)\nu_t = \sum_{j=0}^{\infty} K_j \nu_{t-j} = k(z)\nu_t, \tag{1.4}$$

say, and this is the only solution we consider.

Let rank $(\Sigma) = q$. We say that an ARMA (AR, i.e. $s = 0$) system is *regular* if $q = n$ holds and *singular* if $q < n$. Throughout we assume that q is also specified. The spectral density of (y_t), which is defined on $z \in \mathbb{C}$ rather than on the frequency band $[-\pi, \pi]$, is of the form

$$f_y(z) = \frac{1}{2\pi} a^{-1}(z)b(z)\Sigma b(z^{-1})^T a^{-1}(z^{-1})^T. \tag{1.5}$$

For given spectral density f_y, a transfer function having no poles inside or on the unit circle and no zeros inside is obtained by spectral factorization (see Rozanov (1967); Hannan (1971)). For the regular case such a transfer function is unique if we assume that $B_0 = I_n$. In the singular case we write the system (1.1) as

$$y_t + A_1 y_{t-1} + \cdots + A_p y_{t-p} = b_0 \varepsilon_t + \cdots + b_s \varepsilon_{t-s}, \tag{1.6}$$

where $b_j \in \mathbb{R}^{n\times q}$ and $(\varepsilon_t | t \in \mathbb{Z})$ is white noise with $\mathbb{E}\left(\varepsilon_t \varepsilon_t^T\right) = I_q$. Let $\tilde{b}(z) = \sum_{j=0}^{s} b_j z^j$, then

$$b(z)\Sigma b(z^{-1})^T = \tilde{b}(z)\tilde{b}(z^{-1})^T. \tag{1.7}$$

The matrix $\tilde{b}(z)$ can be written in Smith McMillan form $\tilde{b}(z) = u(z)\Lambda(z)v(z)$ (see e.g. Hannan and Deistler (2012)), where $u(z)$ and $v(z)$ are unimodular polynomial matrices, the top $q \times q$ block of $\Lambda(z)$ is diagonal and all other entries are zero. For general rational matrices $\Lambda(z)$ is rational, whereas in our case, because $b(z)$ is polynomial, $\Lambda(z)$ is polynomial too.

We replace assumption (1.3) by the assumption that $\tilde{b}(z)$ has no zeros inside the unit circle, i.e. the non-zero entries in $\Lambda(z)$ have no zeros inside the unit circle. The transfer function $\tilde{k}(z) = a^{-1}(z)\tilde{b}(z)$, which is a causal and miniphase spectral factor for a given spectral density f_y, is unique up to postmultiplication by constant orthogonal matrices.

The paper consists of two main parts: In the first part, i.e. in Sections 2 and 3, we analyze the structure of singular AR and ARMA systems respectively. In the second part, i.e. in Section 4, we deal with identifiability of the parameters of a high frequency ARMA system from mixed frequency data.

2. Singular AR systems

In this section we present a structure theory for singular AR systems. Many results described here are scattered in recent papers by some of the authors and their coauthors of this paper (Chen et al. (2011); Deistler et al. (2010, 2011); Anderson et al. (2012a)) and in two theses (Filler (2010); Felsenstein (2014)).

A singular AR system can be written as

$$y_t = -A_1 y_{t-1} - \cdots - A_p y_{t-p} + b\varepsilon_t, \tag{2.1}$$

where $A_j \in \mathbb{R}^{n\times n}$ are such that $a(z)$ fulfills the stability assumption (1.2), $b \in \mathbb{R}^{n\times q}$, $q < n$, $\mathbb{E}\left(\varepsilon_t \varepsilon_t^T\right) = I_q$ and $\Sigma = bb^T$. Then b is unique up to postmultiplication by orthogonal matrices.

The parameter space of the AR system (2.1) is the set

$$\Theta_{AR} = \left\{\left(A_1, \ldots, A_p\right) \mid \det a(z) \neq 0, \ |z| \leq 1\right\} \times \{b \mid \operatorname{rank}(b) = q\}. \tag{2.2}$$

The solution of (2.1) is of the form

$$y_t = a^{-1}(z)b\varepsilon_t = \tilde{k}(z)\varepsilon_t \tag{2.3}$$

and the spectral density of (y_t), written as a function of $z \in \mathbb{C}$, is of the form

$$f_y(z) = \frac{1}{2\pi} a^{-1}(z)bb^T a^{-1}(z^{-1})^T. \tag{2.4}$$

Clearly, f_y is singular (as a rational matrix), and

$$\gamma(0) = \mathbb{E}\left(y_t y_t^T\right) = \int_{[-\pi,\pi]} f_y(e^{-i\lambda})d\lambda$$

may be non-singular or singular. The case where $\gamma(0)$ is singular, say of rank r, $q \leq r < n$, may be treated as follows (see e.g. Deistler et al. (2010)): Let

$$\gamma(0) = LL^T,$$

where $L \in \mathbb{R}^{n\times r}$ is unique up to orthogonal postmultiplication. Then we have

$$y_t = Lz_t, \tag{2.5}$$

where

$$z_t = \left(L^T L\right)^{-1} L^T y_t = L^- y_t$$

satisfies

$$\mathbb{E}\left(z_t z_t^T\right) = I_r > 0.$$

Note that, when $\gamma(0)$ is singular, we can always obtain a process with a non-singular variance matrix after a linear static transformation and, consequently, we restrict ourselves to this case of non-singular $\gamma(0)$ from now onwards. Finally, if it was the case that y_t are latent variables in a factor model, then z_t in (2.5) are minimal static factors. In this way the dimensionality of the parameter space is reduced.

2.1. Identifiability and classes of observational equivalence

If we commence from the spectral density (2.4), the transfer function (2.3) is a stable and miniphase spectral factor and is unique up to postmultiplication by (constant) orthogonal matrices under our assumptions. Note that (2.3) is the Wold decomposition: Using the Smith McMillan form

$$\tilde{k}(z) = u(z)\Lambda(z)v(z),$$

we can define a causal left inverse for $\tilde{k}(z)$ as

$$\tilde{k}(z)^- = v(z)^{-1}\left(\Lambda(z)^T\Lambda(z)\right)^{-1}\Lambda(z)^T u(z)^{-1}. \tag{2.6}$$

Because $\tilde{k}(z)$ has no zeros, $\tilde{k}(z)^-$ has no poles and therefore $\tilde{k}(z)^-$ is polynomial.

The Yule Walker equations

$$(\gamma(1), \ldots, \gamma(p)) = \left(-A_1, \ldots, -A_p\right)\Gamma_p \tag{2.7}$$

and

$$bb^T = \gamma(0) + \left(A_1, \ldots, A_p\right)(\gamma(1), \ldots, \gamma(p))^T, \tag{2.8}$$

where

$$\Gamma_m = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(m-1) \\ \gamma(-1) & \gamma(0) & & \\ \vdots & & \ddots & \\ \gamma(-m+1) & & & \gamma(0) \end{pmatrix}, \quad m = 1, 2, \ldots$$

describe the relation between the (population) second moments of (y_t) and the AR parameters $\left(A_1, \ldots, A_p, b\right)$. Note that every solution of (2.7) extends the covariances $\gamma(0), \ldots, \gamma(p)$ in a unique way to the covariance sequence $\gamma(j)$, $j \geq 0$ as

$$\gamma(p+h) = \left(-A_1, \ldots, -A_p\right) \begin{pmatrix} \gamma(p+h-1) \\ \vdots \\ \gamma(h) \end{pmatrix}, \quad h > 0$$

holds.

Also note that in the case of singular AR systems Γ_{p+1} is singular and Γ_p might be singular. When Γ_p is singular, the solution set of (2.7) corresponds to an affine subset for every row of $\left(-A_1, \ldots, -A_p\right)$ corresponding to the shifted kernel of Γ_p. The structure of this set and selecting a unique representative of it are discussed in Deistler et al. (2010); Chen et al. (2011); Deistler et al. (2011). Identifiability analysis for singular AR systems is more involved compared to the regular case and resembles more the ARMA case than to the regular AR case, because a non-trivial common factor problem arises. Note that $(a(z), b)$ is not necessarily left coprime. We have the following result:

Theorem 1. 1. *Let $\tilde{k}(z) = a^{-1}(z)b$, where the pair $(a(z), b)$ satisfies $\det a(z) \neq 0$, $|z| \leq 1$, rank $(b) = q$ and is not necessarily left coprime. Then there exists a left coprime pair $\left(\bar{a}(z), \bar{b}\right)$ satisfying the stability and the rank condition rank $\left(\bar{b}\right) = q$, where in particular $\bar{a}(z)$ has degree at most p, such that $\tilde{k}(z) = \bar{a}^{-1}(z)\bar{b}$.*

2. *For given left coprime $(a(z), b) \in \Theta_{AR}$, such that $\tilde{k}(z) = a^{-1}(z)b$, the class of observationally equivalent AR systems in Θ_{AR} is given by*

$$p(z)\left(a(z), bO\right),$$

where O is an arbitrary constant orthogonal $q \times q$ matrix and $p(z)$ is an arbitrary non-singular $n \times n$ polynomial matrix satisfying

$$\det p(z) \neq 0, \quad |z| \leq 1,$$

$$p(0) = I_n,$$

$$\delta\left(p(z)a(z)\right) \leq p$$

and

$$\delta\left(p(z)b\right) = 0,$$

where $\delta(a(z))$ denotes the degree of the polynomial matrix $a(z)$.

3. *The following statements are equivalent*
   (a) *The system parameters $\left(A_1, \ldots, A_p\right)$ are identifiable and the noise parameters b are identifiable up to a constant orthogonal matrix O, i.e. the parameter set Θ_{AR} contains no observationally equivalent systems except for $(a(z), bO)$.*
   (b) *Γ_p is non-singular.*
   (c) *$(a(z), b)$ is left coprime and rank $\left(A_p, b\right) = n$ holds.*

Proof. For 1. see Anderson et al. (2012a, Theorem 1).

The proof of 2. is straightforward: According to Hannan and Deistler (2012, Theorem 2.2.1) every pair $(\bar{a}, \bar{b})$ observationally equivalent to a left coprime pair (a, b) is related to (a, b) by left multiplication of polynomial matrix $p(z)$. Clearly, from the second moments of (y_t), we are only able to determine bb^T and thus b is only identifiable up to a constant orthogonal matrix O. The condition $\det p(z) \neq 0$, $|z| \leq 1$, has to be fulfilled in order to retain the stability of $p(z)a(z)$. The normalization $p(0) = I$ guarantees $p(0)a(0) = I$. The degree restrictions $\delta\left(p(z)a(z)\right) \leq p$ and $\delta\left(p(z)b\right) = 0$ have to be made to assure that $p(z)\left(a(z), bO\right) \in \Theta_{\mathrm{AR}}$.

Lastly, we prove 3. Obviously, (a)⇔(b) since (2.7) and (2.8) do have a unique solution if and only if $\Gamma_p > 0$. The fact that (a)⇔(c) has already been shown in Hannan (1971). ■

2.2. A state space representation for AR systems

The AR system (2.1) can be written as a state space system of the form

$$\underbrace{\begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} -A_1 & \cdots & -A_{p-1} & -A_p \\ I_n & & & \\ & \ddots & & \\ & & I_n & 0 \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p} \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathcal{B}} \varepsilon_t \tag{2.9}$$

$$y_t = \underbrace{\begin{pmatrix} -A_1 & \cdots & -A_p \end{pmatrix}}_{\mathcal{C}} x_t + \underbrace{b}_{\mathcal{D}} \varepsilon_t. \tag{2.10}$$

In this particular form, the state consists of past outputs. It is easy to see that the following (discrete time) Lyapunov equation is fulfilled

$$\Gamma_p = \mathcal{A}\Gamma_p\mathcal{A}^T + \mathcal{B}\mathcal{B}^T, \tag{2.11}$$

where $\Gamma_p = \mathbb{E}\left(x_t x_t^T\right)$. Because of the stability assumption $\det a(z) \neq 0$, $|z| \leq 1$, the unique solution of (2.11) is

$$\Gamma_p = \sum_{j=0}^{\infty} \mathcal{A}^j \mathcal{B}\mathcal{B}^T \left(\mathcal{A}^j\right)^T.$$

Theorem 2. *The state space system is minimal if and only if $\Gamma_p > 0$ and rank $\left(A_p\right) = n$ hold.*

Proof. Obviously,

$$\Gamma_p = \begin{pmatrix} \mathcal{B} & \mathcal{A}\mathcal{B} & \mathcal{A}^2\mathcal{B} & \cdots \end{pmatrix} \begin{pmatrix} \mathcal{B}^T \\ \mathcal{B}^T\mathcal{A}^T \\ \mathcal{B}^T\left(\mathcal{A}^2\right)^T \\ \vdots \end{pmatrix}$$

holds and thus using the Cayley–Hamilton Theorem we see that $\Gamma_p > 0$ if and only if $(\mathcal{A}, \mathcal{B})$ is controllable. If rank $\left(A_p\right) = n$ holds, $\mathcal{A}$ is non-singular and a right eigenvector $c = \left(c_1^T \ \cdots \ c_p^T\right)^T \neq 0$ of $\mathcal{A}$, $c_i \in \mathbb{R}^n$, corresponding to an eigenvalue $\lambda \neq 0$ has to fulfill

$$-\sum_{i=1}^{p} A_i c_i = \lambda c_1$$

$$c_1 = \lambda c_2$$

$$\vdots$$

$$c_{p-1} = \lambda c_p$$

see e.g. Anderson et al. (2012b, Lemma 2). Thus $c_1 \neq 0$ and obviously c is not in the right kernel of $\left(I_n \ 0 \ \cdots \ 0\right)\mathcal{A}$ and thus $(\mathcal{C}, \mathcal{A})$ is observable. Conversely, if A_p is singular, there exists an eigenvector $c = \left(0 \ \cdots \ 0 \ c_p^T\right)^T$ of $\mathcal{A}$ corresponding to the eigenvalue $\lambda = 0$ which is obviously in the right kernel of $\mathcal{C}$

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and thus $(\mathcal{C}, \mathcal{A})$ is not observable. As minimality is equivalent to controllability and observability, the theorem holds. ■

Remark 1. It should be noted that in some cases a minimal state cannot be constructed from past outputs, with each entry of the state vector corresponding to past outputs, as the following example shows (see Filler (2010); Deistler et al. (2011)): Let

$$y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{5} \end{pmatrix} y_{t-1} + \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix} y_{t-2} + \nu_t$$

be a regular stable AR(2) process ($\mathbb{E}\left(\nu_t \nu_t^T\right) = I_2$) which can be represented as

$$x_{t+1} = \mathcal{A} x_t + \mathcal{B} \varepsilon_t \tag{2.12}$$
$$y_t = \mathcal{C} x_t + \mathcal{D} \varepsilon_t, \tag{2.13}$$

where

$$\mathcal{A} = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{8} & \frac{1}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} I_2 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}, \qquad \mathcal{D} = I_2$$

and

$$x_t = \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix}.$$

A minimal system would be

$$\bar{\mathcal{A}} = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix}, \qquad \bar{\mathcal{B}} = \begin{pmatrix} I_2 \\ 0 \end{pmatrix},$$

$$\bar{\mathcal{C}} = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{8} \end{pmatrix}, \qquad \bar{\mathcal{D}} = I_2$$

and

$$\bar{x}_t = \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{1,t-2} + y_{2,t-2} \end{pmatrix}.$$

This state vector is constructed from past outputs, but does not have each entry corresponding to past outputs.

3. Singular ARMA systems

3.1. The genericity of singular AR systems

The parameter space of the ARMA system (1.6) is the set

$$\Theta_{\text{ARMA}} = \left\{ \left(A_1, \ldots, A_p, b_0, \ldots, b_s\right) \mid \det a(z) \neq 0, |z| \leq 1, \right.$$
$$\left. \operatorname{rank}\left(\tilde{b}(z)\right) = q,\ |z| < 1 \right\}, \tag{3.1}$$

where again $\tilde{b}(z) = \sum_{j=0}^{s} b_j z^j$ is the factorization in (1.7). Assume that $\tilde{b}(z)$ has rank $q < n$ for all $z \in \mathbb{C}$. This is not an unreasonable assumption, in fact we show below that it is generically satisfied. Note that a subset of Θ_{ARMA} is called generic if it contains an open and dense subset. Let

$$\tilde{b}(z) = u(z)\Lambda(z)v(z)$$

denote the Smith McMillan form of $\tilde{b}(z)$. Since $\tilde{b}(z)$ is a polynomial matrix, so is $\Lambda(z)$ and since $\tilde{b}(z)$ has rank q for all $z \in \mathbb{C}$, $\Lambda(z)$ is of the form

$$\Lambda = \begin{pmatrix} I_q \\ 0 \end{pmatrix}.$$

Now $\tilde{b}(z)$ can be extended to a square matrix $b(z)$ as

$$b(z) = u(z) I_n \begin{pmatrix} v(z) & 0 \\ 0 & I_{n-q} \end{pmatrix}$$

which is clearly unimodular. Writing (1.6) as

$$a(z) y_t = b(z) \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix},$$

we obtain

$$b^{-1}(z) a(z) y_t = \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} = \begin{pmatrix} I_q \\ 0 \end{pmatrix} \varepsilon_t \tag{3.2}$$

and (3.2) is a singular AR system as $b^{-1}(z)$ is unimodular (and thus polynomial).

Conversely, if $\left(a(z), \tilde{b}(z)\right)$ are left coprime and if $\tilde{b}(z)$ has zeros, then there is no pseudoinverse of $\tilde{b}(z)$ which is polynomial and thus there exists no AR system generating (y_t).

We will identify $\left(a\left(z\right), \tilde{b}\left(z\right)\right)$ with the corresponding element in Θ_{ARMA}. Whereas for regular ARMA systems, for $s \geq 1$, systems that can be transformed to AR systems form a small subset, for the singular case we have:

Theorem 3. *Let $q < n$. For every s, the polynomial matrix $\tilde{b}(z)$ corresponding to $(b_0, \ldots, b_s) \in \left\{(b_0, \ldots, b_s) \,|\, b_i \in \mathbb{R}^{n \times q}\right\}$, generically has no zeros.*

Proof. First consider the top $q \times q$ submatrix of $\tilde{b}(z)$, $\tilde{b}^q(z)$ say. We will show that generically the zeros of $\det \tilde{b}^q(z)$ are simple and thus the associated kernels of $\tilde{b}^q\,(z)$ are one dimensional.

We use the result that a non-zero scalar function $f : \mathbb{R}^{q^2(s+1)} \rightarrow \mathbb{R}$ which is polynomial in $\operatorname{vec}\left(b_0^q, \ldots, b_s^q\right) \in \mathbb{R}^{q^2(s+1)}$ is generically nonzero, see e.g. Bochnak et al. (1998). We apply this result to the determinant of the Sylvester matrix of $\det \tilde{b}^q(z)$ and its derivative, see Kailath (1980). The entries not restricted to zero of this Sylvester matrix are the coefficients of $\det \tilde{b}^q(z)$ and its derivative and thus the determinant of this Sylvester matrix is a polynomial function in the coefficients of $\det \tilde{b}^q(z)$ and its derivative. Clearly, a zero of $\det \tilde{b}^q(z)$ has multiplicity larger than one if and only if it is also a zero of the derivative of $\det \tilde{b}^q(z)$ and thus is a zero of the determinant of this Sylvester matrix. The proof is completed if we can find one $\tilde{b}^q(z)$ such that the determinant of the Sylvester matrix is nonzero. Such an example can easily be found if we consider a diagonal polynomial matrix $\tilde{b}^q(z)$ with b_s^q non-singular. In this case $\det \tilde{b}^q(z) = \prod_{i=1}^{q} \tilde{b}_{ii}(z)$ and we can easily find a suitable $\tilde{b}^q(z)$. Thus we have shown that generically the zeros $z_1, \ldots, z_{sq}$ of $\det \tilde{b}^q(z)$ are simple and the corresponding kernels are one dimensional.

If we consider the $(q+1)$th row of $\tilde{b}(z_i)$, we see that generically this row will not be orthogonal to the kernel of $\tilde{b}^q(z_i)$. This will hold for all z_i, $i = 1, \ldots, sq$, and since the intersection of a finite number of generic sets is again generic this concludes the proof. This proof is analogous to the proof given in Felsenstein (2014) for state space systems. ■

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A similar result has been derived in Anderson and Deistler (2008); Filler (2010). Let

$$\Theta_{\text{ARMA, LC}} = \left\{ (A_1, \ldots, b_s) \in \Theta_{\text{ARMA}}, \left(a(z), \tilde{b}(z) \right) \text{ is left coprime} \right\}. \quad (3.3)$$

Corollary 1. *Generically in $\Theta_{\text{ARMA, LC}}$, the corresponding process (y_t) has an (singular) AR representation of the form (3.2).*

Proof. Now, with the same argument as in the proof of Theorem 3, an arbitrary $\left(a(z), \tilde{b}(z) \right) \in \Theta_{\text{ARMA}}$, not necessarily left coprime, can be shown to be generically of full rank n for all $z \in \mathbb{C}$. This property is equivalent to $\left(a(z), \tilde{b}(z) \right)$ being left coprime, see e.g. Hannan and Deistler (2012, Lemma 2.2.1). This proves the corollary. ■

3.2. Identifiability

Analogously to the result derived in Hannan (1971) for the regular case we have:

Theorem 4. *The subset of $\Theta_{\text{ARMA, LC}}$, where $\operatorname{rank}\left(A_p, b_s\right) = n$ holds, is identifiable up to postmultiplication of $\tilde{b}(z)$ by arbitrary orthogonal matrices.*

From the results above it is clear that the set of all identifiable systems (up to postmultiplication by arbitrary orthogonal matrices) is generic in Θ_{ARMA}.

3.3. Kernel representation

By our assumptions, the spectral density f_y is singular (as a rational matrix) of rank q. Consider an arbitrary rational $(n-q) \times n$ matrix $w(z)$ whose rows form a basis for the left kernel of f_y. As $w(z)$ is unique only up to premultiplication by non-singular rational matrices, $w(z)$ can be chosen as polynomial and even left coprime. As is easily seen,

$$w(z) f_y(z) = 0$$

is equivalent to

$$w(z) y_t = 0.$$

Since $w(z)$ is left coprime there exists a $(n-q) \times (n-q)$ submatrix, $w_1(z)$ say, which is non-singular in a neighborhood of $z = 0$ and thus has a causal (but not necessarily stable) inverse. After a conformal reordering of variables of y_t we write

$$y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$$

and

$$w(z) = (w_1(z), w_2(z)).$$

Since $y_{1,t}$ is stationary

$$y_{1,t} = -w_1(z)^{-1} w_2(z) y_{2,t} \quad (3.4)$$

defines an exact and causal relation between the output $y_{1,t}$ and the input $y_{2,t}$. Note that, in general, such a selection of inputs and outputs may not be unique.

4. g-identifiability of the parameters of a high frequency ARMA system from mixed frequency data

Here we are concerned with a "high frequency" ARMA system of the form (1.1), together with the assumption (1.2), (1.3) and $B_0 = I_n$. For the sake of simplicity, we restrict ourselves to the regular case, i.e. $q = n$; however, as will be seen below, the results can easily be extended to the singular case. We call (1.1) the high frequency system because it generates the outputs y_t for every $t \in \mathbb{Z}$. Note that, under our assumptions, $k(z) = a^{-1}(z) b(z)$ and Σ are uniquely determined from f_y by spectral factorization.

Unless the contrary is stated explicitly, throughout in this paper we will assume that the MA order s satisfies

$$s \leq p. \quad (4.1)$$

Clearly, this is a restriction of generality and it excludes, for instance, MA systems (with $s \geq 1$). The restriction is triggered by the fact that our main results on identifiability from mixed frequency data require this condition. As it is shown in Example 1, even for a simple MA(1) system, generic identifiability cannot be achieved.

The parameter space considered here is of the form

$$\begin{aligned} \Theta = & \left\{ \left(A_1, \ldots, A_p, B_1, \ldots, B_p \right) \mid \det a(z) \neq 0, |z| \leq 1, \right. \\ & \left. \det b(z) \neq 0, |z| < 1 \right\} \left\{ \Sigma \mid \Sigma = \Sigma^T, \Sigma > 0 \right\}. \end{aligned} \quad (4.2)$$

As far as identifiability of the parameters from the "high frequency process" $(y_t \mid t \in \mathbb{Z})$ (i.e. all outputs can be observed) is concerned (see e.g. Hannan (1971); Hannan and Deistler (2012)) the subset

$$\begin{aligned} \Theta_I = & \left\{ \left(A_1, \ldots, B_p, \Sigma \right) \in \Theta \mid \operatorname{rank}\left(A_p\right) = n, \right. \\ & \left. (a(z), b(z)) \text{ is left coprime} \right\} \end{aligned} \quad (4.3)$$

is identifiable. For the sake of simplicity we will restrict ourselves to this setting.

Generalizations of the results in this paper to the case of prescribed column or row degrees are straightforward. Note that Θ_I is an open and dense subset of Θ since $\operatorname{rank}\left(A_p\right) = n$ as well as left coprimeness are open and dense properties, the latter is proved in the proof of Corollary 1.

Identifiability of high frequency parameters in the AR case from mixed frequency data is discussed in detail, both for the regular and the singular case, in Anderson et al. (2015). In this contribution we extend (some of) these results to the ARMA case.

As it is well known, (1.1) can be transformed into a state space form and minimal state space systems are unique only up to basis change. The assumption (4.1), $\operatorname{rank}\left(A_p\right) = n$ and $(a(z), b(z))$ is left coprime imply that the dimension of a minimal state is equal to np (compare Hannan and Deistler (2012), chapter 2). Here we consider a unique state space representation of the form (compare Kailath (1980); de Jong and Penzer (2004))

$$x_{t+1} = F x_t + G \nu_t \quad (4.4)$$

$$y_t = H x_t + \nu_t, \quad (4.5)$$

where

$$F = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & 0 & I_n \\ -A_p & -A_{p-1} & & \cdots & -A_1 \end{pmatrix} \quad (4.6)$$

$$G = \begin{pmatrix} K_1 \\ \vdots \\ K_p \end{pmatrix}, \quad H = \begin{pmatrix} I_n & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $k(z) = H \left(I_{np} - Fz \right)^{-1} Gz + I_n$ and

$$a(z) k(z) = b(z) \quad (4.7)$$

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implies

$$\begin{pmatrix} I_n & 0 & \cdots & 0 \\ A_1 & I_n & \cdots & 0 \\ \vdots & & \ddots & \\ A_p & A_{p-1} & \cdots & I_n \end{pmatrix} \begin{pmatrix} I_n \\ K_1 \\ \vdots \\ K_p \end{pmatrix} = \begin{pmatrix} I_n \\ B_1 \\ \vdots \\ B_p \end{pmatrix}. \tag{4.8}$$

We consider the case where the data are stock variables. Let y_t be partitioned as

$$y_t = \begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix}, \tag{4.9}$$

where the n_f-dimensional high frequency vector y_t^f is observed for every $t \in \mathbb{Z}$ and the n_s-dimensional low frequency vector y_t^s is observed only for $t \in N\mathbb{Z}$ (N being an integer larger than one). Throughout we assume that $n_f \geq 1$.

The central problem considered in this section is *identifiability*, i.e. whether, for given Θ_I, the parameters A_i, B_i and Σ of the high frequency system are uniquely determined by those population second moments which can be directly observed. These population second moments are of the form

$$\gamma^{ff}(h) = \mathbb{E}\left(y_{t+h}^f \left(y_t^f\right)^T\right), \quad h \in \mathbb{Z};$$

$$\gamma^{sf}(h) = \mathbb{E}\left(y_{t+h}^s \left(y_t^f\right)^T\right), \quad h \in \mathbb{Z};$$

$$\gamma^{ss}(h) = \mathbb{E}\left(y_{t+h}^s \left(y_t^s\right)^T\right), \quad h \in N\mathbb{Z}. \tag{4.10}$$

As will be shown, generic identifiability can be obtained under our assumptions, i.e. on an open and dense subset of Θ_I and thus of Θ. We call this g-identifiability.

Example 1. We consider a regular MA(1) system, with $n = 2$, which can be written, using an obvious notation, as

$$\begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix} = \begin{pmatrix} \nu_t^f \\ \nu_t^s \end{pmatrix} + B_1 \begin{pmatrix} \nu_{t-1}^f \\ \nu_{t-1}^s \end{pmatrix}. \tag{4.11}$$

Before understanding the difficulties caused by lack of data, let us consider what happens when all covariances are available. Observing the MA equation, we see that there are apparently 7 parameters, 4 for the matrix B_1 and 3 for the matrix Σ. To understand how these relate to the spectrum, let us observe that this spectrum necessarily has the form

$$f_y(z) = \gamma(1)z + \gamma(0) + \gamma(1)^T z^{-1}. \tag{4.12}$$

It immediately follows that

$$\Sigma + B_1 \Sigma B_1^T = \gamma(0), \quad B_1 \Sigma = \gamma(1).$$

One can regard the task of finding the MA model as one of solving these equations for the entries of B_1, Σ given $\gamma(0)$ and $\gamma(1)$. There are in general a finite number of solutions, including one with the miniphase property, i.e. $\det b(z) \neq 0$ for $|z| < 1$. The other solutions have one or more determinantal zeros which are reflections through the boundary of the unit circle of the determinantal zeros of the minimum phase solution.

Now in the mixed frequency case, further ambiguities arise. The second moments which can be directly obtained from the data comprise the following covariances for the case $N = 2$: $\gamma(0)$, $\gamma^{ff}(1)$, $\gamma^{sf}(1)$ and $\gamma^{fs}(1)$. With the exception of $\gamma^{ss}(1)$, which corresponds to the $(2, 2)$ entry of $\gamma(1)$ and cannot be directly observed, all other covariances are zero because of the MA(1) structure. Now, as easily can be shown, $\gamma^{ss}(1)$ can be varied without violating the positive definiteness of the covariance sequence. The corresponding variation of the MA parameters (all satisfying the miniphase condition) shows that we have non-identifiability for every observed MA(1) covariance in this case.

4.1. The relation between parameters and second moments of the outputs. Yule Walker type equations

We start with the following lemma:

Lemma 1. *The covariances* $\gamma(h) = \mathbb{E}\left(y_h y_0^T\right)$ *are of the form*

$$\gamma(h) = HF^{h-1}\left(FPH^T + G\Sigma\right), \quad h \geq 1 \tag{4.13}$$

$$\gamma(0) = HPH^T + \Sigma, \tag{4.14}$$

where $P = \mathbb{E}\left(x_t x_t^T\right)$. *They are rational functions of the parameters* $\theta \in \Theta_I$.

Proof. From (4.4) we obtain

$$P = FPF^T + G\Sigma G^T \tag{4.15}$$

and by columnwise vectorizing this equation we get

$$\mathrm{vec}(P) = \left(I_{(np)^2} - F \otimes F\right)^{-1} \mathrm{vec}\left(G\Sigma G^T\right),$$

compare Anderson et al. (2015). This shows that P is a rational function of θ.

Now, for $h \geq 1$, it follows that

$$\begin{aligned} \gamma(h) &= \mathbb{E}\left(y_h y_0^T\right) = \mathbb{E}\left((Hx_h + \nu_h)(Hx_0 + \nu_0)^T\right) \\ &= \mathbb{E}\left(H\left(F^h x_0 + \sum_{i=0}^{h-1} F^{h-1-i} G\nu_i\right)(Hx_0 + \nu_0)^T\right) \\ &= HF^h PH^T + HF^{h-1} G\Sigma \end{aligned}$$

and $\gamma(0) = HPH^T + \Sigma$. Thus it is easy to conclude that $\gamma(h)$ is rational. ■

We now consider the relation between the ARMA parameters and the population second moments of the output process (i.e. not only those second moments which can be observed). Postmultiplying (1.1) by y_{t-p-j}^T, $j > 0$, we obtain the following Yule Walker type equations for the AR parameters

$$(\gamma(p+1), \gamma(p+2), \ldots) = \left(-A_1, \ldots, -A_p\right) \underbrace{\begin{pmatrix} \gamma(p) & \gamma(p+1) & \cdots \\ \gamma(p-1) & \gamma(p) & \cdots \\ \vdots & \vdots & \\ \gamma(1) & \gamma(2) & \cdots \end{pmatrix}}_{\Gamma_{p,\infty}}. \tag{4.16}$$

Now, using the results in Hannan and Deistler (2012, chapter 2), the matrix $\Gamma_{p,\infty}$ can be shown to have rank np for all $\theta \in \Theta_I$: From Theorem 2.4.1 in Hannan and Deistler (2012) the dimension of the minimal state in (4.4), (4.5) which is np is equal to the rank of the infinite dimensional Hankel matrix

$$\mathcal{H}_{\infty,\infty}^k = \begin{pmatrix} K_1 & K_2 & K_3 & \cdots \\ K_2 & K_3 & K_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of the transfer function. As is easily shown the rank of $\mathcal{H}_{\infty,\infty}^k$ is equal to the rank of

$$\Gamma_{\infty,\infty} = \begin{pmatrix} \gamma(1) & \gamma(2) & \cdots \\ \gamma(2) & \gamma(3) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

From (4.16) it is immediate that $\mathrm{rank}\left(\Gamma_{p,\infty}\right) = np$ must hold. From the Hankel structure we have $\mathrm{rank}\left(\Gamma_{p,\infty}\right) = \mathrm{rank}\left(\Gamma_{p,np}\right)$, where

$\Gamma_{p,np}$ consists of the first np block columns of $\Gamma_{p,\infty}$. Thus we have from (4.16)

$$\begin{aligned}&\left(-A_1, \ldots, -A_p\right)\\ &= \left(\gamma\left(p+1\right), \ldots, \gamma\left(p(n+1)\right)\right)\Gamma_{p,np}^T\left(\Gamma_{p,np}\Gamma_{p,np}^T\right)^{-1}. \qquad (4.17)\end{aligned}$$

Thus for all $\theta \in \Theta_I$, $\left(A_1, \ldots, A_p\right)$ is uniquely determined from the second moments of the outputs $(y_t \mid t \in \mathbb{Z})$. Then $\left(B_1, \ldots, B_p\right)$, as well as Σ are uniquely obtained from the spectral factorization of

$$f_b(z) = a(z)f_y(z)a(z^{-1})^T, \qquad (4.18)$$

see e.g. Rozanov (1967); Hannan (1970).

4.2. The relation between the high frequency parameters and the second moments of the observed outputs

Here we commence from the second moments of the observed outputs, see (4.10). For the AR case, i.e. when $s = 0$ holds in (1.1), we have (see Chen and Zadrozny (1998), Anderson et al. (2015))

$$\begin{aligned}&\mathbb{E}\left(y_t\left((y^f_{t-1})^T, \ldots, (y^f_{t-np})^T\right)\right)\\ &= \left(-A_p, \ldots, -A_1\right)\underbrace{\mathbb{E}\left(\begin{pmatrix} y_{t-p}\\ \vdots \\ y_{t-1}\end{pmatrix}\left((y^f_{t-1})^T, \ldots, (y^f_{t-np})^T\right)\right)}_{Z_{\mathrm{AR}}}, \qquad (4.19)\end{aligned}$$

where $Z_{\mathrm{AR}} \in \mathbb{R}^{np \times np n_f}$. It is shown in Anderson et al. (2015) that Z_{AR} has generically (i.e. on a set containing a subset that is open and dense) in the $\left(A_1, \ldots, A_p, \Sigma\right)$ parameters rank equal to np. In the same way the extended Yule Walker equations for our ARMA case can be written as

$$\begin{aligned}&\mathbb{E}\left(y_t\left((y^f_{t-p-1})^T, (y^f_{t-p-2})^T, \ldots, (y^f_{t-p(n+1)})^T\right)\right)\\ &= \left(-A_p, \ldots, -A_1\right)\\ &\quad\times \underbrace{\mathbb{E}\left(\begin{pmatrix} y_{t-p}\\ \vdots \\ y_{t-1}\end{pmatrix}\left((y^f_{t-p-1})^T, (y^f_{t-p-2})^T, \ldots, (y^f_{t-p(n+1)})^T\right)\right)}_{Z_{\mathrm{ARMA}}}. \qquad (4.20)\end{aligned}$$

Now, Z_{ARMA} is a rational function of $\theta \in \Theta_I$ by Lemma 1 and thus $\det\left(I_{(np)^2} - F \otimes F\right) Z_{\mathrm{ARMA}}$ is a polynomial function of $\theta \in \Theta_I$. By the stability assumption $\det\left(I_{(np)^2} - F \otimes F\right) \neq 0$ on Θ_I and thus the zeros of corresponding subdeterminants of Z_{ARMA} and of $\det\left(I_{(np)^2} - F \otimes F\right) Z_{\mathrm{ARMA}}$ are the same. As has been shown in Anderson et al. (2015), there exists a $\theta^* \in \Theta_I$ corresponding to the case $s = 0$ such that Z^*_{AR} has full rank np. Since $Z^*_{\mathrm{ARMA}} = F^p Z^*_{\mathrm{AR}}$ and F has full rank np, Z^*_{ARMA} has also full rank np. Then for a suitable chosen $np \times np$ submatrix of Z^*_{ARMA}, its determinant is unequal to zero and thus for general Z_{ARMA} the corresponding determinant is unequal to zero except for a proper variety on Θ_I (see Bochnak et al. (1998), page 23). Therefore, in particular, Z_{ARMA} has full row rank on a generic subset of Θ_I. Thus we have shown:

Theorem 5. *Generically the parameters $\left(A_1, \ldots, A_p\right)$ are uniquely obtained from (4.20). They are g-identifiable on Θ_I and on Θ.*

Now, for given $\left(A_1, \ldots, A_p\right)$ and $\gamma(N)$, $\gamma(2N)$, $\gamma(3N)$, ..., we have from (4.13)

$$\begin{aligned}\begin{pmatrix}\gamma\left(N\right)\\ \gamma\left(2N\right)\\ \gamma\left(3N\right)\\ \vdots\\ \gamma\left(npN\right)\end{pmatrix} &= \begin{pmatrix}HF^{N-1}\\ HF^{2N-1}\\ HF^{3N-1}\\ \vdots\\ HF^{npN-1}\end{pmatrix}\left(FPH^T + G\Sigma\right)\\ &= \mathcal{O}^N\left(FPH^T + G\Sigma\right), \qquad (4.21)\end{aligned}$$

where F and H are known (F being given from (4.6) and (4.20)). Note that by minimality and thus observability of (4.4), (4.5), for $N = 1$, the matrix $\mathcal{O}^N$ has full column rank np. It is straightforward to show that the assumption that all eigenvalues of F^N are distinct is generic in Θ_I (see Felsenstein (2014); Koelbl (2015)). Under this assumption the eigenvectors of F and F^N are the same and then $\operatorname{rank}\left(\mathcal{O}^N\right) = \operatorname{rank}\left(\mathcal{O}^1\right)$ holds, as it can be shown by the PBH test (see Kailath (1980)). Multiplying (4.21) from the left by the pseudoinverse $\left(\left(\mathcal{O}^N\right)^T\mathcal{O}^N\right)^{-1}\left(\mathcal{O}^N\right)^T$ we obtain $\left(FPH^T + G\Sigma\right)$. Using (4.13) we obtain all second moments $\gamma\left(0\right), \gamma\left(1\right), \ldots$ from $\left(A_1, \ldots, A_p\right)$ and $\gamma\left(N\right), \gamma\left(N\right), \ldots, \gamma\left(npN\right)$, and therefore the MA parameters are unique under our assumptions. Thus we have shown:

Theorem 6. *For a generic subset of Θ_I, the ARMA parameters θ are uniquely determined from the second moments (4.10) of the observed outputs and thus are g-identifiable on Θ_I.*

Note that the same argument can be used also for the singular case, as long as Z_{ARMA} has rank np. The only difference then is that we have to replace $B_0 = I_n$ by a more subtle normalization.

5. Conclusions

This paper deals with the structure of multivariate AR and ARMA systems in two non-standard cases. We consider singular AR or ARMA systems. We derive identifiability results for the singular case and we show that a process, which is generated by a singular ARMA system, has generically a singular AR representation.

The second focus of this paper is on identifiability results for high frequency ARMA systems in the case of mixed frequency observations, where the slow output is observed for every Nth time point. The main result is that, under the assumption that the MA order is smaller than or equal to the AR order, identifiability of the system and noise parameters can be achieved for a generic subset of the parameter space. This paper generalizes the results obtained by Anderson et al. (2015) for the AR case to the ARMA case. Furthermore, we give an example for the pure MA case where the system and noise parameters are not even generically identifiable.

A detailed analysis of identifiability for singular ARMA systems as well as an analysis for the case of flow data or more general aggregation schemes is left to future research.

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