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A Brief View of Strange Attractors

061 **Key Words**
062 Dynamic system
063 Chaos
064 Strange attractors
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Abstract

The chaos theory has been one of the outstanding recent developments in mathematics, since (1) it shows how seemingly simple systems can model most complex ('chaotic') behaviors and (2) it is constructive in the sense that it allows a quantitative numerical analysis of time series paving the way for numerous applications in many disciplines, including medicine.

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For almost 10 years now mathematics and the natural sciences have been riding a wave which, in its power, creativity and expanse has become an interdisciplinary experience of the first order. For some time now this wave has also been touching distant shores far beyond the sciences. Never before have mathematical insights – usually seen as dry and dusty – found such rapid acceptance and generated so much excitement in the public mind. Fractals and chaos have literally captured the attention, enthusiasm and interest of a worldwide public. To the casual observer, the color of their essential structures and their beauty and geometric form captivate the visual senses as few other things they have ever experienced in mathematics. And to the scientist, fractals and chaos offer a rich environment for exploring and modelling the complexity of nature.

In this short note an exposition of just a few aspects of the chaos theory is presented, which can be studied in more detail elsewhere [1]. For further information a list of books covering chaos theory and its applications is supplied [1–20]. Here we restrict ourselves to chaos in dynamic systems of two or more dimensions. This is the relevant case for models in the natural sciences since very rarely processes can be described by only one single state variable. One of the main players in this context is the notion of *strange attractors.*

To talk about strange attractors we have to consider a particular kind of dynamic systems: dissipative dynamic systems, i.e. systems with some sort of friction. The chief feature of dissipative systems is loss of energy. For example, a real pendulum swinging in air will have dissipation. Energy is lost continuously through the various kinds of friction which the pendulum experiences. In contrast, we speak of conservative dynamic systems when energy is maintained. This is the case in systems without friction. For example, the friction which heavenly bodies sustain is so little that we think of their motion as conservative; no energy is lost.

Guided by methematical development, physicists and mathematicians were led to believe that the long-term behavior of dissipative systems would always run into simple patterns of motion such as a rest point or a limit cycle. In contrast, strange attractors are those patterns which characterize the final state of dissipative systems that are highly complex and show all the signs of chaos. They very strongly defy the power of an intuitive understanding, and yet they now are proven to be all around us. It seems as if all of a sudden a whole new world of previously invisible beings is flying around us. Moreover, strange attractors are the point where chaos and fractals meet in an unavoidable and most natural fashion: as geometrical patterns, strange attractors are fractals; as dynamic objects, strange attractors are chaotic. There is now a whole new branch of experimental and theoretical science dealing with strange attractors, their classification, the measurements of their quantitative properties, their reconstruction from physical data, and so on. But undoubtedly the mathematical understanding of strange attractors is just in its infancy and it will be one of the great challenges of future mathematical generations.

It is by no means easy to understand even the notion of a strange attractor. In fact, strange attractors still have not received a final mathematical definition. Mathematics is sometimes described as the science which generates eternal notions and concepts for the scientific method: derivatives, continuity, powers and logarithms are examples. The notions of chaos, fractals and strange attractors are not yet mathematical notions in that sense, because their final definitions are not yet agreed upon.

The first strange attractor ever recognized as such in the natural sciences is the Lorenz attractor, discovered in 1962. However, the work was published in the *Journal of the Atmospheric Sciences* which is not usually read by physicists and mathematicians. So the research on chaos was unnecessarily delayed by a decade or so until the real implications of Lorenz's achievement became clear. Although the Lorenz attractor is one of the 'oldest' known strange attractors, answers to some very basic questions about it are still outstanding.

Given a dynamic system such as the Lorenz system, we can see the attractors on our computer graphics screens. This is fine. However, when physicists, for example, make measurements in some real-world experiment, they only obtain long and messy sequences of numbers, not equations. Then they must answer the question what kind of dynamic system is behind the scene or perhaps even whether there is a strange attractor lurking behind their irregular and noisy data. One of the most fascinating achievements of chaos theory is that it has made available a tractable numerical method to attack this problem, the *reconstruction of strange attractors.* It even leads to algorithms which can compute numerical quantities such as dimensions and Ljapunov exponents that specify the degree of strangeness and 'chaoticity' of the attractor.

In 1976 Rössler found a particularly simple system, which is probably the most elementary geometric construction of chaos in continuous systems. Thus, before we start to discuss the Lorenz attractor, let us follow Rössler's ideas. His system of differential equations is

$$x' = -(y + z)$$
$$y' = x + ay \quad (1)$$
$$z' = b + xz - cz$$

where the three coefficients a, b, c are adjustable constants.

We can interpret the system in equation 1 as a collection of laws of motion for a point at coordinates (x y, z) in a three-dimensional system. For any given initial coordinates (x_o, y_o, z_o) the system defines a unique trajectory which is parametrized by time t and satisfies the equations at all times. Denoting the coordinates of this trajectory by [x(t), y(t), z(t)] for time t ≥ 0 means

$$x'(t) = -[y(t) + z(t)] \qquad x(0) = x_o$$
$$y'(t) = x(t) + ay(t) \qquad y(0) = y_o$$
$$z'(t) = b + x(t)z(t) - cz(t) \qquad z(0) = z_o.$$

Figure 1 shows a plot of three components x(t), y(t), z(t) versus time t. However, it is more instructive to plot this trajectory in three-dimensional space with coordinates (x, y, z) and consecutive points connected by short line segments; we obtain a first picture of the Rössler attractor (fig. 2). Orbits on the attractor spend most of their time near the xy-plane spiraling out from the origin. When an orbit has attained some critical distance from the origin it is first lifted away from the xy-plane. Then, after reaching some maximal z-value, it is reinserted into the spiraling piece of the attractor close to the plane. The larger the z-amplitude of this excursion has been, the closer to the origin the orbit will land, and the spiraling process followed by ejection and reinsertion repeats.

Let us first take a look at the phenomenology of the attractor by means of some simple numerical experiments before we try to understand how the equations provide the foundations for these effects. First of all we note that we are indeed dealing with an attractor. When we start the solution of the differential equation at some other initial point somewhere in the vicinity of the structure shown in figure 2, we get essentially the same result. Only the first part of the trajectory is noticeably different stemming from the transitional period necessary for the solution to get close to the attractor.

In order to understand the chaotic behavior of the dynamics in Rössler's attractor, we begin by showing how the nonlinear stretch-and-fold operation is hidden in the system. From the pictures, it appears that the attractor has the structure of a *folded band*. Starting from a section across the band near the negative x-axis we can observe two effects. (1) As the band winds around the center for about half a turn its width increases. This corresponds to the stretching. (2) Near the positive x-axis the band is more than twice as wide and the outer part begins to fold over and eventually covers the inner part and also part of the 'hole' at the center. This folding action is completed after about another half of a turn and the process then repeats.

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The stretch-and-fold operation can be uncovered in the following experiment. We define a cross-section of the Rössler band as a reference frame – for example the part of the attractor which lies along the negative x-axis where the x-values of trajectories are at their minimum. The band is quite flat, and therefore we can identify points in the intersection of the attractor and the half-plane simply by the absolute values of their x-coordinates. We regard such points as initial points and follow their trajectories around the band for one complete turn until they reenter the half plane. We arrive at a new position with a new x-coordinate. In order to record the stretch-and-fold operation we can mark in a diagram the absolute value of this new x-value compared to the old one. When performing this procedure for a long trajectory on the Rössler band, we should obtain a good representation of the stretch-and-fold dynamics, which is in the same spirit as the parabola in the graphical iteration method (fig. 4). And moreover, the figure indeed reveals a shape which closely resembles a parabola! A function modeling the plot is called a *Lorenz map*. It provides a link between the dynamics of a continuous system and the discrete dynamics of transformations of an interval. Once this link is known, it provides a shortcut for computing the dynamics of the underlying system. Instead of following a trajectory of the differential equation with possibly many steps and a large computational effort, we may simply evaluate the Lorenz map once (or perform one step of graphical iteration) and arrive at the same result. What is more, it is usually much easier to analyze the properties of a Lorenz map than the dynamics for a differential equation. Thus, the results of the theory regarding the chaos for transformations of intervals and the routes leading to it help to understand the dynamics of the more complicated continuous system.

The Rössler system is an artificial system designed solely with the purpose of creating a model for a strange attractor which uses only the simplest chaos-generating mechanism, stretch and fold. Of course, Rössler knew about the Lorenz system, which had been published 13 years before. In fact, we may say the Rössler attractor is a model of the Lorenz model.

The system of equations that Lorenz proposed does not look any more complicated than that of Rössler. Here it is:

$$
\begin{aligned}
x' &= -\sigma x + \sigma y \\
y' &= Rx - y - xz \\
z' &= -Bz + xy.
\end{aligned} \tag{2}
$$

σ, B, and R are the physical parameters of the system, which Lorenz fixed at

$$\sigma = 10,\ B = \frac{8}{3},\ R = 28.$$

Figure 5 shows the corresponding attractor, which is now called the Lorenz attractor. Clearly, the geometry is more involved than for the Rössler band. There are two sheets in which trajectories spiral outwards. When the distance from the center of such a spiral becomes larger than some particular threshold, the solution is ejected from the spiral and attracted by the other spiral, where it again begins to spiral out and the game is repeated. The number of turns that a trajectory spends in one spiral and then in the other is not specified. It may wind around one spiral two times, then three times around the other, then ten times around the first and so on. In fact, we believe that for any sequence of positive numbers which are not too large, for example 3, 11, 7, ... there exists a trajectory on the Lorenz attractor with precisely these numbers as turns around the spirals. Thus, there is a solution that turns 3 times around the right spiral, then 11 times around the left, then 7 times around the right again and so on.

What is the connection between these wildly spinning solutions and weather forecasting which is what Lorenz was interested in? Certainly, the trajectories should not be mistaken for the paths of air currents! If this were the case then the Lorenz attractor would act similar to a black hole in astrophysics, sucking in all the atmosphere – leaving nothing but emptiness around it and laying waste to the whole planet Earth. But we are not far from the truth. The Lorenz system is in fact a model of thermal convection which, however, includes not only a description of the motion of some viscous fluid or atmosphere but also the information about distribution of heat, the driving force of thermal convection.

When air is warmed near the Earth's surface it rises. This is an important factor in the atmospheric weather factory. Convection air currents may accumulate and give rise to convection cells of several types and, when forced more vigorously, may produce very turbulent motion in the atmosphere. Examples of convection cells are cylindrical rolls and structures, which are called *Bénard* cells, resembling a honeycomb from above. In these hexagonal cells the warmed portions of the fluid rise in the center, get colder near the top and sink back down to the surface around the boundary of the cell. The Lorenz system is related more to the cylindrical roll-type of fluid motion in which one of the dimensions can be disregarded pretending that these rolls extent to infinity. The mathematical model of the fluid motion had originally been developed by Lord Rayleigh in 1916. It assumes that all convection happens in a rectangular region whose bottom is heated such that the temperature difference between bottom and top remains constant. With certain parameter configurations in this model it turns out that the solutions to the model equations have a rather special form which was already known to Rayleigh. Lorenz took these special solutions, regarded their amplitudes as time dependent, inserted them in the Rayleigh model, disregarded all terms that are not in this special form, and arrived at a system of differential equations for the time-dependent amplitudes (equation 2).

Understanding natural processes does not start with a set of equations for a dynamic system. On the contrary, such models are usually obtained at the end of a long course of action consisting of the identification of the phenomena to be studied, conducting series of often difficult and elaborate experiments, running trial and error computer simulations, and finally making a mathematical analysis. Somewhere in this process the question about order or chaos arises. How can we determine from measured data whether there is some underlying deterministic governing equation for the phenomena observed, or whether the data are merely noise without any structure? In other words, we want to know from a given sequence of numbers whether they come from an attractor, and if so, we also need to quantify this attractor in terms of Ljapunov exponents and dimension so that we may speak of a chaotic and strange attractor.

Let us imagine a somewhat simpler situation; we have a black box in which some continuous dynamic system is running. We may only probe the system at discrete time intervals and obtain the value of one of the state variables of the system. For example, choose one of the variables, calling it z(t), and a time interval τ. Our examination of the black box would yield the sequence of numbers

$$z_0 = z(0), z_1 = z(\tau), z_2 = z(2\tau), z_3 = z(3\tau), \dots \quad (3)$$

Given such data, can we reconstruct some meaningful picture of some underlying attractor? Can we say something about its dimension and the Ljapunov exponents? At first thought this seems a rather hopeless effort.

The prospects, however, are not as bleak as they seem. Here is an example. Figure 6 shows three time series all of which look more or less random. However, if one of them really is deterministic, then the numbers must follow some rule. In other words, z_k may be determined from the past of the sequence, and we may hope to be able to put it in the form of

$$z_k = \psi(z_{k-1}, z_{k-2}, z_{k-3},...)$$

where ψ denotes some (so far) unknown transformation. Let us be even more optimistic and assume that z_k strongly depends on its predecessor, z_{k-1}, and only mildly (or not at all) on all earlier predecessors. To check this assumption we produce plots of z_k versus z_{k-1} as shown in figure 7. The result is quite clear. There is no evident structure in the first set of data. The points obtained from the second series clearly lie on a section of what appears to be a parabola. This tells us that this set of data can be generated by means of graphical iteration of a function whose graph is the parabola pictured in figure 7. In fact, we used the formula $4x(1-x)$ of the generic parabola to produce the data. Thus, this simple procedure already enables us to completely unravel the random-looking data and to uncover its deterministic quadratic generation process. But will such a cheap trick work in real applications where other variables are hidden in the 'black box' or the dependence of the presence on the past is more complicated? The third data set presents such a case. The corresponding plot of z_k versus z_{k-1} shows a collection of points which are not distributed throughout the entire square in contrast to the points of the first random data set. They form a clear structure which, however, cannot be obtained as the graph of a function. But this structure can be interpreted as an attractor of some underlying system which is a crucial insight opening the door to further numerical investigations.

Thus, this method of analysis of time series may lead to useful results. The fact is that a straight forward extension of this very simple procedure allows us to retrieve the geometric structure of *any* underlying attractor. Let us hide the Rössler system in the 'black box', run the system, and again assume that the sequence in equation 3 is extracted from the machine. We now choose a time delay T (a multiple of τ) and look at the following sequence of vectors

$$[z(0), z(T), z(2T)]$$
$$[z(\tau), z(\tau + T), z(\tau + 2T)]$$
$$[z(2\tau), z(2\tau + T), z(2\tau + 2T)]$$
$$\vdots$$
$$[z(k\tau), z(k\tau + T), z(k\tau + 2T)].$$

Plotting these points in three-dimensional space with connecting line segments, we obtain figure 8. Clearly the essential features of the Rössler attractor are apparent.

This is not an accident! Strange attractors theoretically can always be faithfully reconstructed using the above procedure. However, working in three dimensions, we cannot expect the procedure to perform when the dimension of the attractor surpasses 3. In such a case a dense subset of the three-dimensional space would be filled. We may, however, simply work in higher-dimensional spaces using vectors.

$$u(t) = [z(t), z(t + T), \ldots, z(t + 2NT)]$$

with 2N + 1 components. IF N is chosen large enough the attractor will 'fit' in the chosen space. Following some theorems derived by Ricardo Mañé and Floris Takens, this can be guaranteed if the dimension of the attractor is not larger than N. Here the choice of the time lag T is almost arbitrary. However, in practice there are limitations. If T is quite small, then the vectors to be plotted will have components which are almost identical, resulting in a reconstructed attractor which will be very close to the 'diagonal' of the space. On the other hand if T is very large, then there is only very little correlation between the components of the vectors, and trajectories on the attractor appear to wander all around phase space such that the structure is hard to detect.

The reconstruction of strange attractors can be interpreted as a change of coordinates. Often the attractor is defined in some infinite-dimensional space (e.g. a space of functions). In this case the reconstruction amounts to a projection of the original to a finite-dimensional euclidean space. Choosing the dimension 2N + 1 of the embedding space large enough guarantees that the projection is injective. This means that each point in the projected attractor corresponds to one and only one point in the original attractor. In other words, we see a truthful representation and not some image where parts of the attractor are collapsed onto each other. Thus, the reconstructed attractor is not identical to the original but a more or less distorted copy. Changing coordinates moderately does not effect the dimension or the Ljapunov exponents. Thus, we should be able to extract that information from a time series of a single variable.

Soon after the news of strange attractors had spread around the scientific laboratories of the world in the 1970s researchers became aware of the subject and concentrated on the irregular patterns of processes which they previously had dismissed as misfits. For example, reconstructions of strange attractors have been successfully completed in a number of applications ranging from ultrasonic cavitation to cardiology. Without doubt, chaos theory has become an integral part of the scientific studies in many disciplines and is here to stay, even after the current enormous public interest in chaos theory will have tapered off.

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Fig. 1. A trajectory. The initial condition (–1, 0, 0) for Rössler's system (equation 1) produces these plots of x(t), y(t) and z(t) versus time t. The parameters are a = b = 0.2, c = 5.7.

Fig. 2. The Rössler attractor. The trajectory from figure 1 is plotted in three-dimensional space revealing a first picture of the Rössler attractor. Two projections are given in figure 3.

Fig. 3. The Rössler attractor. Top view (left) and side view (right).

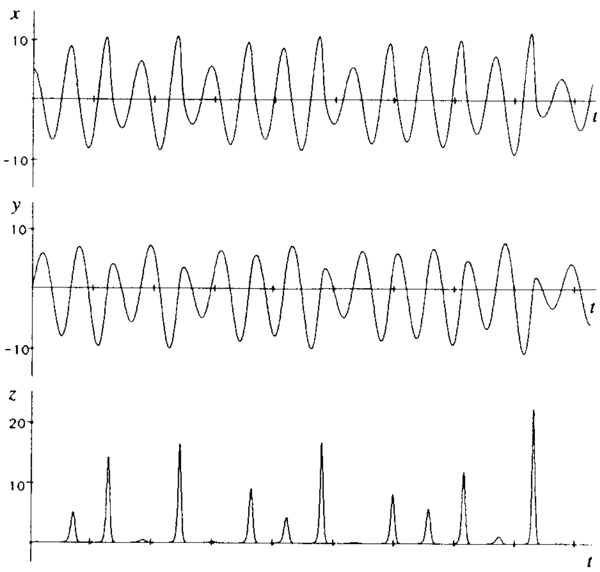
Fig. 4. The Lorenz map for Rössler's attractor. A trajectory has local minima of x-values near the negative x-axis. After one turn around the attractor the next minimum is attained. The graph above the attractor, the Lorenz map of this system, displays this new x-value as a 'function' of the old one.

Fig. 5. The Lorenz attractor. Some trajectories from the Lorenz attractor.

Fig. 6. Random or not random data? Three series obtained from a 'black box'. Which one is random, which is deterministic? Or are they all random?

Fig. 7. Reconstruction attempts. The plots show the points z_k versus z_{k-1} from the three time series in figure 6. This test shows that the first series is apparently random, while the second comes from a simple one-dimensional deterministic system. In the third, there is some structure indicating a strange attractor.

Fig. 8. Reconstruction of the Rössler attractor. Using a time delay of T = 0.5 we obtain a picture of phase space with a reconstruction of the Rössler attractor.



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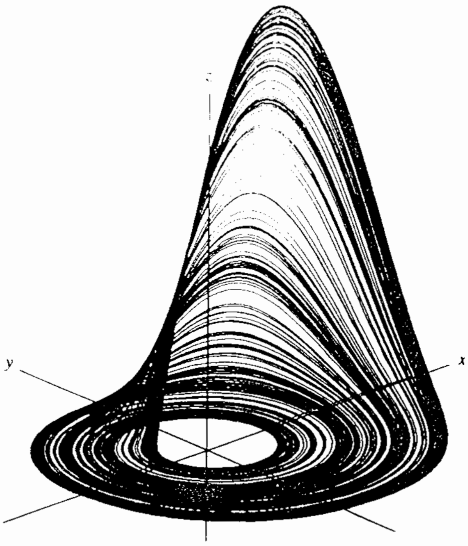

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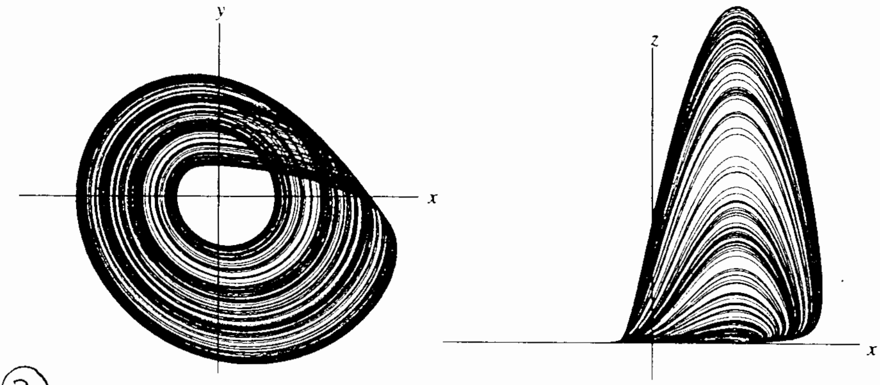

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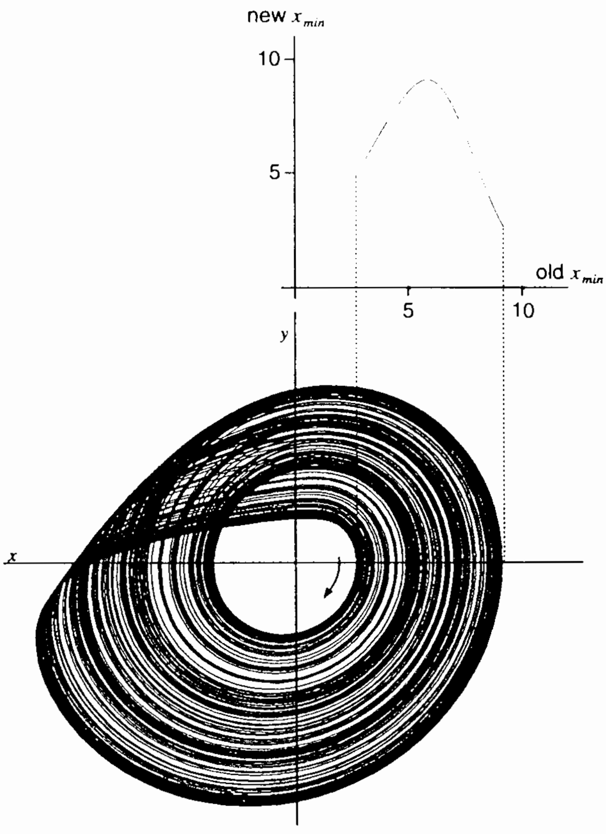

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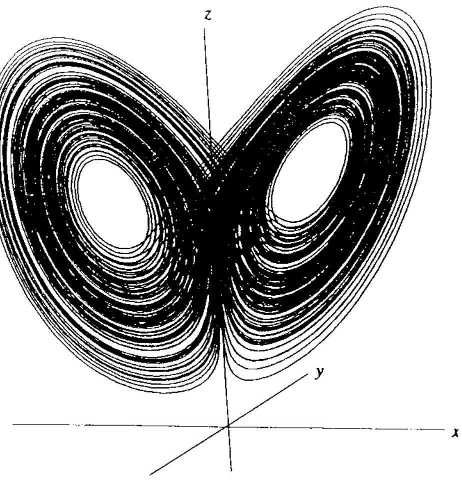

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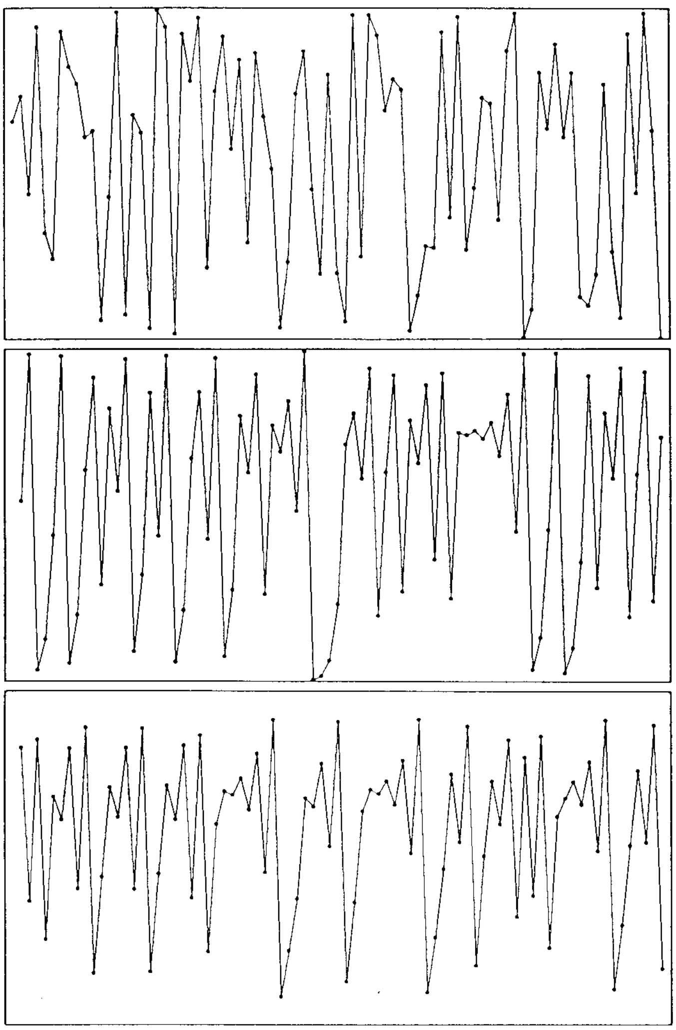

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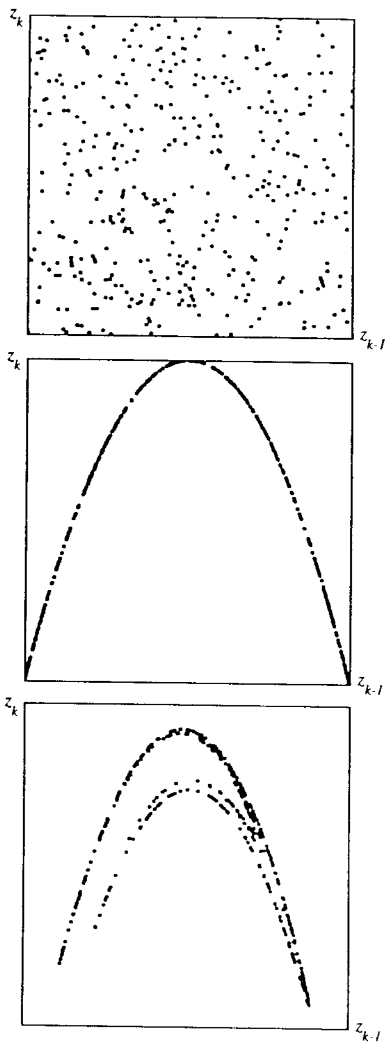

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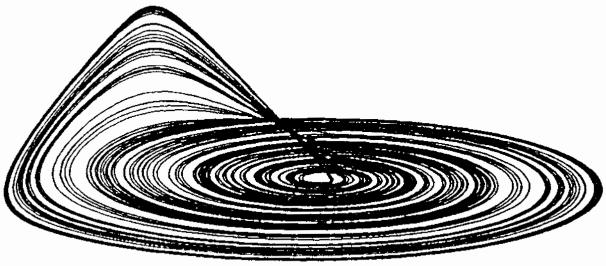

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