The Language of Fractals

These unimaginably detailed structures are more than mathematical curiosities. Fractal geometry succinctly describes complex natural objects and processes

by Hartmut Jürgens, Heinz-Otto Peitgen and Dietmar Saupe

"Nature has played a joke on mathematicians. The 19th-century mathematicians may have been lacking in imagination, but nature was not. The same pathological structures that the mathematicians invented to break loose from 19th-century naturalism turned out to be inherent in familiar objects all around us."

—FREEMAN DYSON
"Characterizing Irregularity,"
Science, May 12, 1978

The "pathological structures" conjured up by 19th-century mathematicians have, in recent years, taken the form of fractals, mathematical figures that have fractional dimension rather than the integral dimensions of familiar geometric figures (such as one-dimensional lines or two-dimensional planes). The current fascination with fractals is largely a result of the work of Benoit B. Mandelbrot of the IBM Thomas J. Watson Research Center in Yorktown Heights, N.Y. Mandelbrot coined the term fractal in 1975; he derived the word from the Latin *fractus*, the adjectival form of *frangere*, or "to break." The concept of fractals exploded into the consciousness of mathematicians, scientists and the lay public in 1983, when Mandelbrot's ground-breaking book, *The Fractal Geometry of Nature*, was published.

Fractals are much more than a mathematical curiosity. They offer an extremely compact method for describing objects and formations. Many structures have an underlying geometric regularity, known as scale invariance or self-similarity. If one examines these objects at different size scales, one repeatedly encounters the same fundamental elements. The repetitive pattern defines the fractional, or fractal, dimension of the structure. Fractal geometry seems to describe natural shapes and forms more gracefully and succinctly than does Euclidean geometry.

Scale invariance has a noteworthy parallel in contemporary chaos theory, which reveals that many phenomena, even though they follow strict deterministic rules, are in principle unpredictable. Chaotic events, such as turbulence in the atmosphere or the beating of a human heart, show similar patterns of variation on different time scales, much as scale-invariant objects show similar structural patterns on different spatial scales. The correspondence between fractals and chaos is no accident. Rather it is a symptom of a deep-rooted relation: fractal geometry is the geometry of chaos.

Another parallel between fractal geometry and chaos theory lies in the fact that recent discoveries in both fields have been made possible by powerful modern computers. This development challenges the traditional con-

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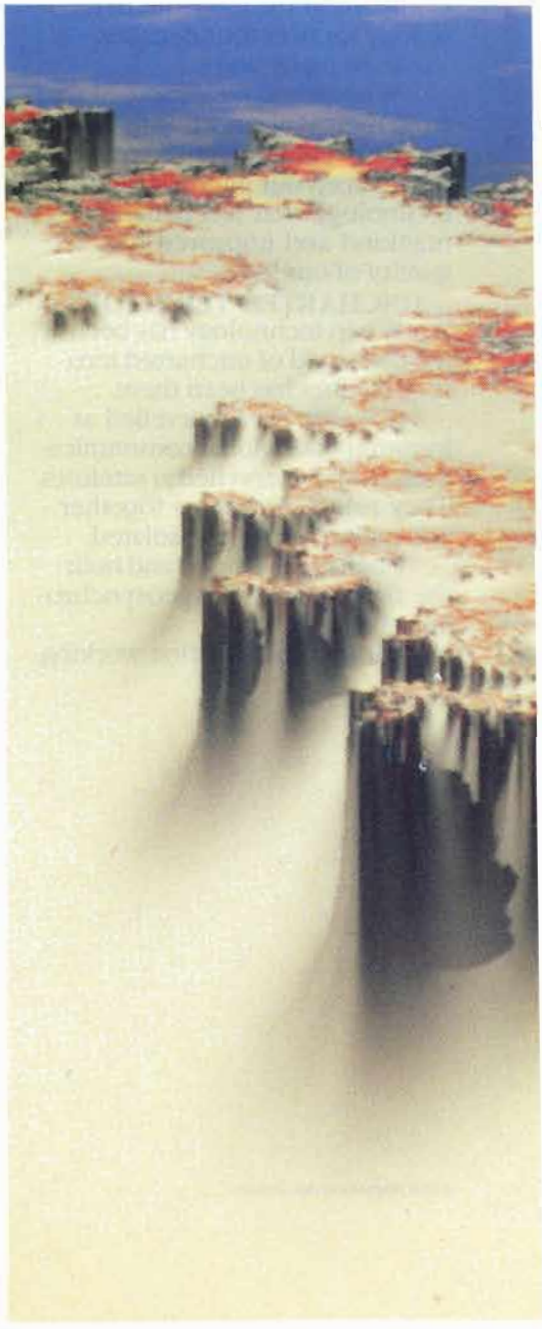


THREE-DIMENSIONAL renderings of the Mandelbrot set have been used to study this fascinating, complex fractal structure. The map shows the electric potential surrounding an electrically charged Mandelbrot set. The eerie similarity between the Mandelbrot set and features in the real world points to the prevalence of fractallike structures throughout nature. The image is taken from a computer-animated videotape produced by the authors and their colleagues.

ception of mathematics. Many mathematicians have greeted computers with a sense of rejuvenation and liberation, but others view them as a rejection of pure mathematics.

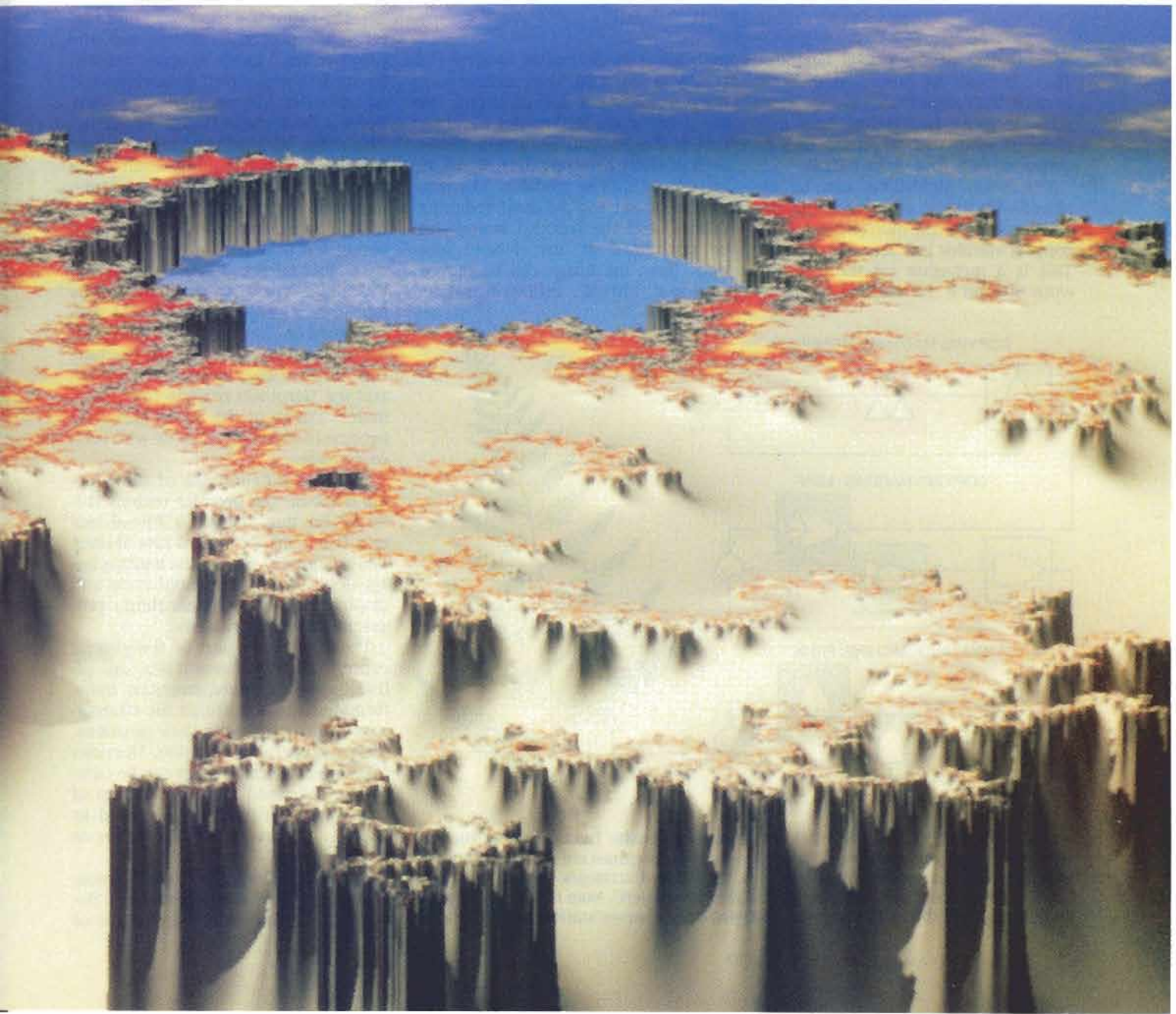
Fractals are first and foremost a language of geometry. Yet their most basic elements cannot be viewed directly. In this aspect they differ fundamentally from the familiar elements of Euclidean geometry, such as the line and circle. Fractals are expressed not in primary shapes but in algorithms, sets of mathematical procedures. These algorithms are translated into geometric forms with the aid of a computer. The supply of algorithmic elements is inexhaustibly large. Once one has a command of the fractal language, one can describe the shape of a cloud as precisely and simply as an architect might describe a house with blueprints that use the language of traditional geometry.

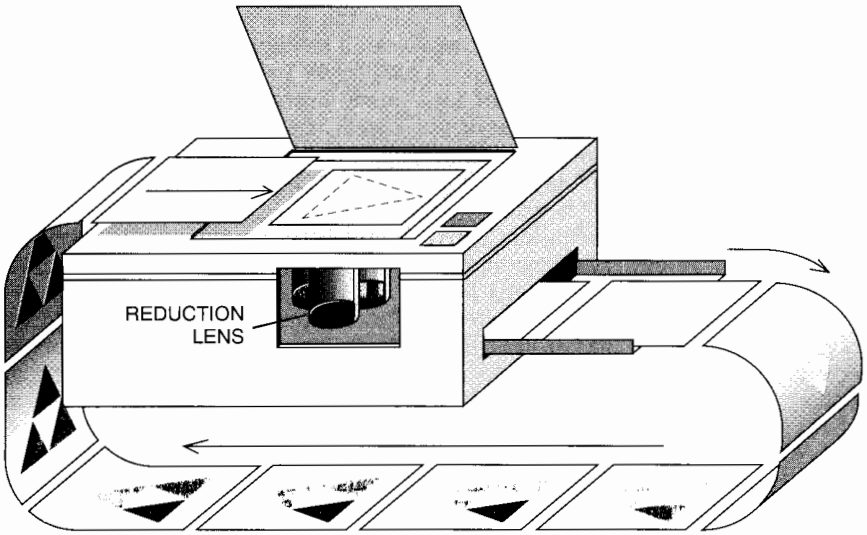
Language is an apt metaphor for the ideas that underlie fractal geometry. Indo-European languages are based on a finite alphabet (the 26 letters from which English words are constructed, for instance). Letters do not carry meaning unless they are strung together into words. Euclidean geometry likewise consists of only a few elements (line, circle and so on) from which complex objects can be constructed. These objects, in a sense, only then have geometric meaning.

Asian languages such as Mandarin Chinese are made up of symbols that themselves embody meaning. The number of possible symbols or elements in these languages is arbitrarily large and can be considered infinite. Fractal geometry is constructed in much the same way. It is made up of infinitely many elements, each complete and unique. The geometric elements are defined by algorithms, which function as units of "meaning" in the fractal language.

There are two main fractal language groups: linear and nonlinear. Both are spoken using an infinite number of algorithms and therefore encompass an infinite number of possible fractal images. The language of nonlinear fractals is far richer and more varied, however. Most dialects follow a deterministic set of rules (analogous to spelling and grammar). One family of fractals, called random fractals, differs in that it is assembled by incorporating controlled randomness.

Linear fractal geometry is the most





MULTIPLE-REDUCTION COPYING MACHINE performs a feedback loop that creates a fractal form. Several lenses transform an arbitrary initial image (input) into a new image (output) that is a collage of reduced copies of the input. The output image is itself run through the machine over and over, producing a final image.

basic dialect of the fractal language. These fractals are described as linear because their algorithms have the same form as those that define lines in a plane (in mathematical jargon, they incorporate only first-order terms).

The linear algorithms can be explored with the aid of an imaginary image duplicator—the multiple-reduction copying machine [*see illustration above*]. This is a metaphor for the beautiful work of John E. Hutchinson, a mathematician at the Australian National University in Canberra. The machine functions much like a normal copier that has a reduction option but differs in that it has several reduction lenses, each of which can copy the original image placed on the machine. The lenses can be set for different reduction factors, and the reduced images can be positioned at any desired location. Thus, the image can be moved, stretched, shrunk, reflected, rotated or transformed in any way, as long as the straight lines of the image remain straight.

The manner in which the image is repositioned and reduced is determined by the algorithm. A feedback loop processes the image over and over, gradually building up a fractal form. One example of a fractal produced by a feedback (recursive) algorithm is the Sierpinski triangle, named after the Polish mathematician Waclaw Sierpinski, who first described it in 1916. The Sierpinski triangle is self-similar: every part of the figure, no matter how small, contains an image that can be magnified to produce the entire Sierpinski triangle.

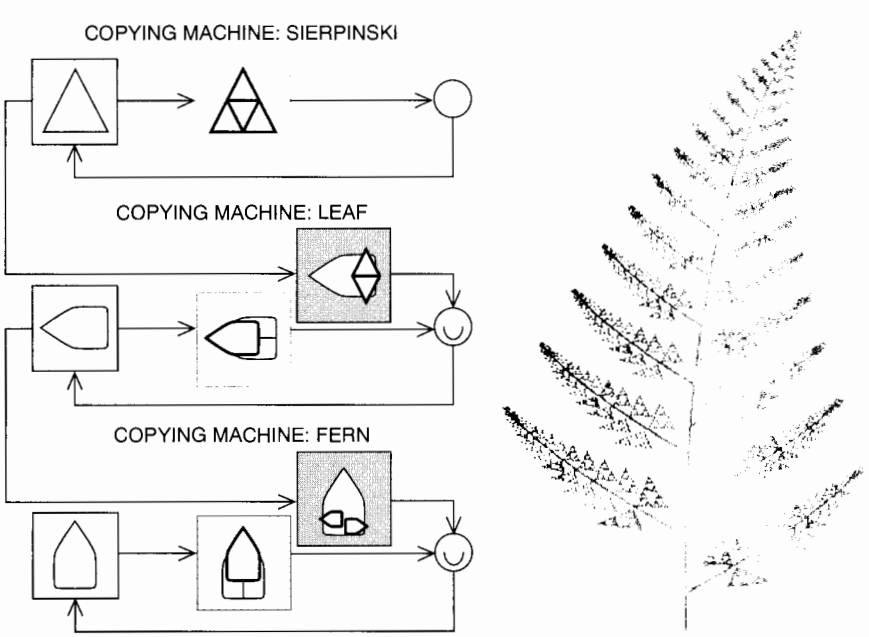
The Sierpinski triangle is created in a multiple-reduction copying machine in the following manner. An image is placed on the machine, reduced by one half and copied three times, once onto each vertex of an equilateral triangle. The result is a triad configuration. When the procedure is repeated, the previous triad image is again reduced by one half and copied three times and so on. After just six copies, or iterations, a final shape begins to appear. This shape is called the limiting image because it is the limiting result of an infinite number of cycles of the copying machine. The limiting image can be quickly approximated but can never be completely achieved.

The limiting image is independent of the initial image. A distinctive initial image, such as the word FRACTAL, can be placed on the copying machine. After six copying runs in the machine the initial image is nearly invisible, and the Sierpinski triangle form dominates. Traces of the initial FRACTAL become increasingly obscure with each copy made.

Small reconfigurations of the copying machine can produce entirely different limiting images: a fractal tree or a frond-shaped fractal [*see illustration on opposite page*]. The limiting image depends only on the reduction and displacement rules (algorithms) programmed into the machine.

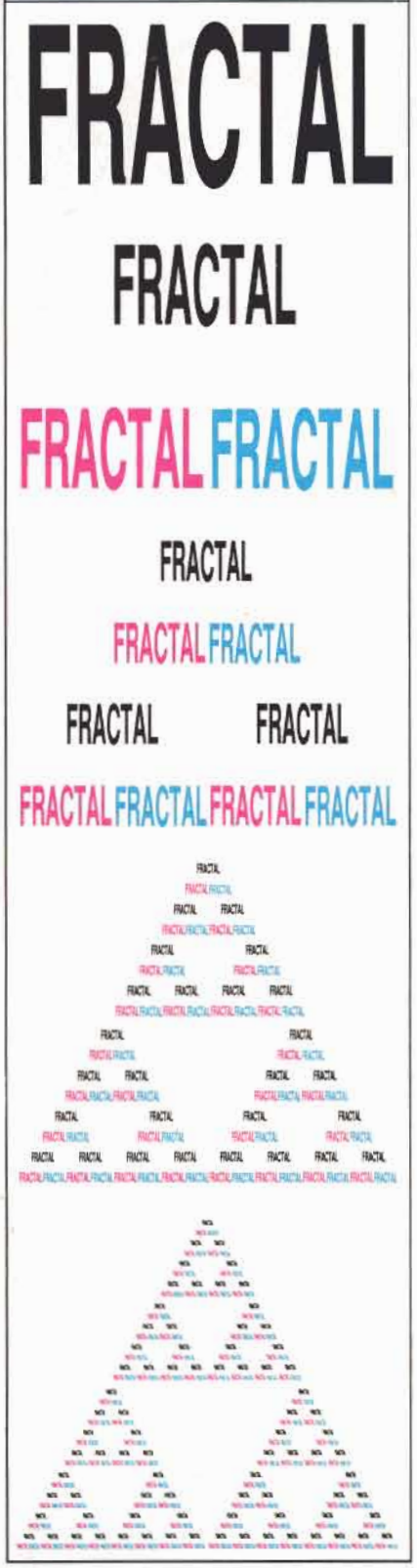
These rules are part of the general concept that mathematicians call affine-linear plane transformations: transformations that maintain the straightness of lines but alter their positions, scale and overall orientation. The rules for a linear fractal dialect can be completely described by a number (n) of transformation functions, denoted as $\{f_1, f_2, \ldots f_n\}$ [*see top of illustration on opposite page*].

Herein lies one of the great practical potentials of fractal geometry. Describing suitable objects by means of

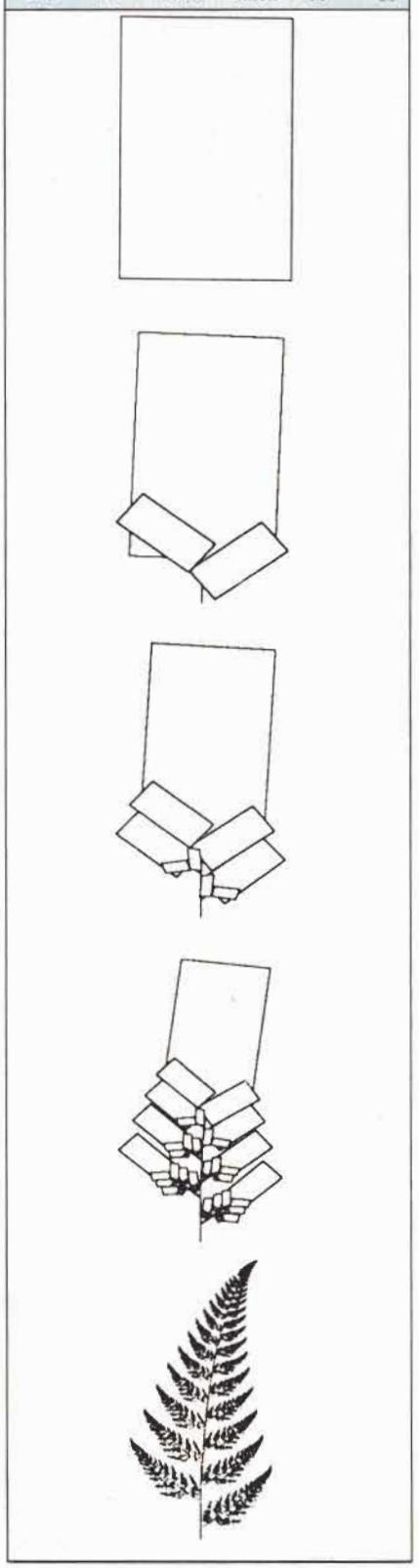


NETWORKED COPYING MACHINE can create composite fractal images such as this fern composed of Sierpinski triangles. Several machines are linked together in parallel; one creates Sierpinski triangles, the second arranges the triangles into leaves and the third produces the overall fern shape (*left*). Note that leaves branch alternatingly off the main stem; the triangles on the leaves stand opposite (*right*).

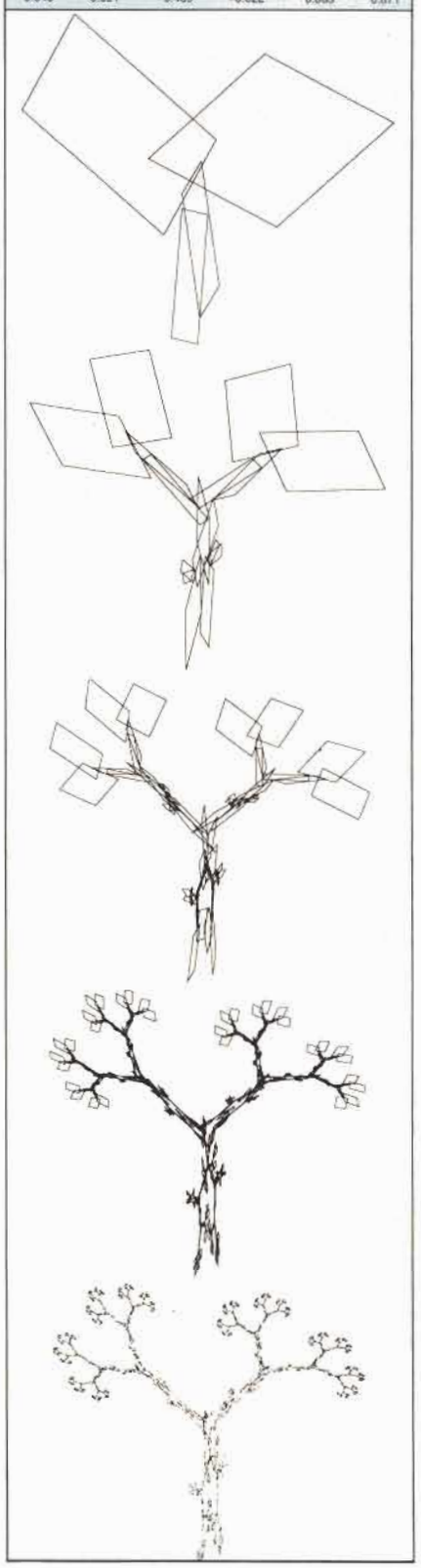
a_{11}	a_{12}	a_{21}	a_{22}	b_1	b_2
0.5	0.0	0.0	0.5	0.0	0.0
0.5	0.0	0.0	0.5	0.5	0.0
0.5	0.0	0.0	0.5	0.25	0.5



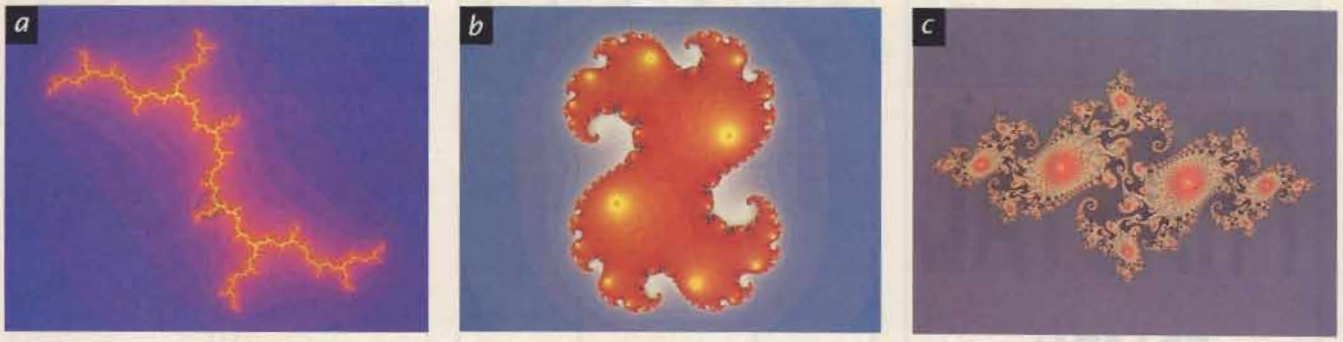
a_{11}	a_{12}	a_{21}	a_{22}	b_1	b_2
0.0	0.0	0.0	0.17	0.0	0.0
0.84962	0.0255	−0.0255	0.84962	0.0	3.0
−0.1554	0.235	0.19583	0.18648	0.0	1.2
0.1554	−0.235	0.19583	0.18648	0.0	3.0



a_{11}	a_{12}	a_{21}	a_{22}	b_1	b_2
0.195	−0.488	0.344	0.443	0.722	0.536
0.462	0.414	−0.252	0.361	0.538	1.167
−0.058	−0.070	0.453	−0.111	1.125	0.185
−0.045	0.091	−0.469	−0.022	0.863	0.671



FRACTAL IMAGES from the feedback loop of the copying machine depend only on the programmed copying routine. The word FRACTAL is transformed by a program that reduces an image to one half its size and copies it three times, once at each corner of an equilateral triangle. The resulting image is known as a Sierpinski triangle (*left*). Slightly more elaborate transformations of this kind result in a fern-shaped fractal (*center*) or a fractal tree (*right*). Any initial image would produce the same result when put through the copying machine. A few numbers defining the copying rules (*top*) are sufficient to describe an image that would require hundreds of thousands of numbers to describe by conventional means.



JULIA SETS are fractal boundaries that emerge from the iteration of the quadratic transformation z^2+c. They take on a bewildering variety of shapes that depend only on the number c, called a control parameter. Some values of c lead to Julia

a linear fractal dialect can significantly reduce the amount of data necessary to transmit or store an image. This is convincingly demonstrated by the fern frond. A complex form like a fern can be completely described by a linear algorithm that is based on only 24 numbers! In contrast, representing the image of the leaf point for point at television-image quality would require several hundred thousand numerical values. In principle, any image may be coded using the appropriate set of linear transformation functions.

The time, complexity and cost of transmitting satellite images to the earth could be drastically reduced by converting them into codes using fractal algorithms. Such a possibility raises a crucial and still largely unsolved problem. How does one obtain the smallest possible family of transformation functions $\{f_1, \ldots f_n\}$ necessary to define an image to the desired precision? The problem currently is the focus of intense study. More general applications of such a procedure would include codes to create halftone or even color images.

Fractal-image coding is only useful if some efficient method exists for extracting the image locked away in the fractal algorithms. The fractal fern offers a good opportunity to examine how the image is produced. The copying-machine rules for this fractal specify that every transformation results in four reductions and repositionings of the previous image. One transformation performs a particularly rapid reduction that squeezes the image into a vertical line; this line forms the stem.

If one begins with a single rectangle, the number of rectangles increases fourfold from copy to copy, totaling 4^m after m transformations. After four iterations the initial image (in this case, a rectangle) is still easily identifiable. For the rectangle to be small enough so that the limiting image (the fern) is visible, one would have to perform roughly 50 iterations and therefore calculate and draw 4^{50} (approximately 10^{30}) rectangles. The task would exceed the capacity of any existing computer.

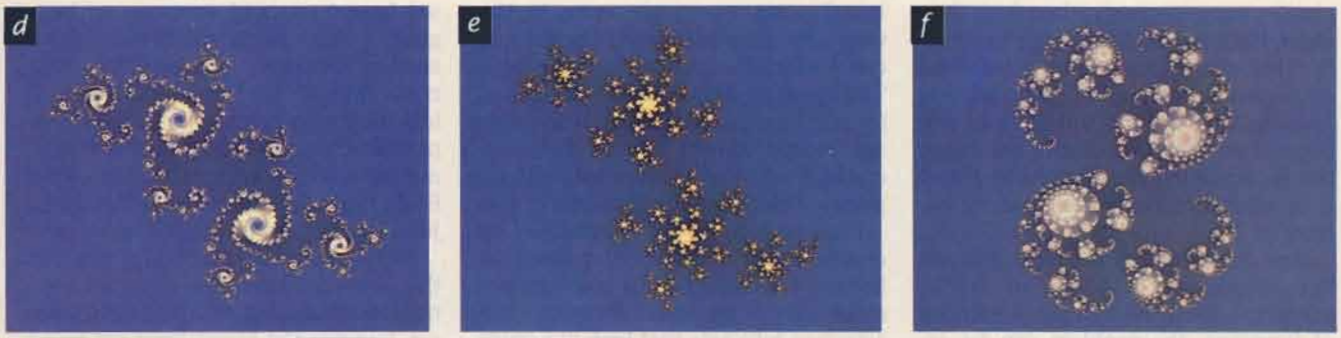
Given this difficulty, one might wonder how these limiting images are produced. The trick that makes the creation of these images possible is an algorithm we call the chaos game, which was suggested by Michael E. Barnsley and Stephen Demko of the Georgia Institute of Technology. The game begins with the selection of an arbitrary point on a plane. Next a four-sided die, each side of which corresponds to one of the four transformations that create the fern figure, is thrown. The roll of the die randomly calls up one of the transformations $\{f_1, f_2, f_3, f_4\}$, which is then applied to the marked point, moving it to a new point on the plane. Another roll selects another transformation, which is applied to the point previously obtained and so on. The points produced by successive throws of the die soon settle down and densely fill up the limiting image. The problem with this technique is that it may take an extremely long time to obtain this image.

In the above example the roll of the die gives equal probability to each f_k (k simply represents one of the possible functions). The limiting image can be obtained much faster if each f_k is assigned a probability P_k with which it is thrown in the chaos game so that certain f_k's become more likely than others. An image can be formed fastest by assigning the greatest probability to the functions that reduce the image the least. This alteration causes the chaos game to hit each spot of the limiting image equally often, with the result that all parts of the image fill in equally quickly.

Altering the chaos game makes it possible to describe halftones simply by linking the frequency with which an image point is hit to a gray-scale value. Through the appropriate choice of P_k, a desired gray-scale value (in other words, the desired frequency of hits) can be obtained for every image point. Applying this technique to the additive primary colors (red, green and blue) permits color images to be coded. Thus, the value of fractal data compression is enhanced even further.

So far there is no satisfactory method for automatically generating fractal encodings of a given picture or image. For self-similar images such as the Barnsley fern there is a semiautomatic procedure that involves interaction be-

NONLINEAR FRACTALS such as Julia sets also can be created by an appropriately designed multiple-reduction copying machine. The lenses no longer perform simple reductions; they now bend and distort the images in addition to shrinking them.



sets that are in one piece (*left*). Other values yield Julia sets that are disconnected and dustlike (*right*). The Mandelbrot set consists of all points *c* associated with connected Julia sets; it also functions as a table of contents of Julia sets.

tween computer and viewer. First one breaks down the image into parts that are similar to the whole image. In the case of the fern the two lower leaflets are similar in form to the whole, as is the upper part of the fern that is left when the lower leaflets are removed. A multiple-reduction copying machine can be designed to incorporate transformations that reduce the whole image into these parts. This can be done easily by trial and error using an interactive computer program.

The method's underlying concept suggests that only strictly self-similar images can be coded in fractal form. This restriction can be overcome by means of a promising extension of the method, which is currently being researched. The central idea is to operate several copying machines simultaneously in parallel in a hierarchical network. Such a network can control individual self-similar features or mix more than one. It becomes possible, for instance, to create a fernlike leaf composed of Sierpinski triangles [*see illustration on page 62*].

We now turn to a different set of fractal dialects, the nonlinear dialects. One of these, the quadratic dialect, has garnered particular attention. It produces a great wealth of geometric forms from a fairly simple algorithm, and it is closely related to current chaos theory.

The theory behind the quadratic dialect was first described in 1918 by the French mathematician Gaston Julia while he was in a military hospital recovering from injuries suffered during World War I. Both his work and the contemporaneous work of his ardent competitor, Pierre Fatou, were soon all but forgotten, but recently Mandelbrot's work has revitalized interest in their theory. The intellectual achievements of Julia and Fatou are particularly remarkable because these men had no computers at their disposal and therefore had to rely solely on their own inherent powers of visualization.

Julia and Fatou were interested in complex numbers, each of which consists of a real number and a multiple of i, the imaginary number defined as the square root of -1. Complex numbers are commonly plotted in a plane along two perpendicular axes, one of which represents real numbers, the other imaginary ones. The two men were trying to resolve what happens to a sequence of points z_k in the complex-number plane that are generated by the transformation $g(z) = z^2 + c$. A new point, z_{k+1}, is obtained by putting the preceding point, z_k, through the transformation. The complex number c is a control parameter that can be chosen at will. This seemingly simple feedback process forms the basis for a dazzling array of shapes.

When an initial point z_0 is put through the transformation, the resulting sequence behaves one of two ways. Either it roams freely, increasing toward infinity, or it is trapped within a certain region of the complex-number plane. The unfettered points are known as the escape set; those that remain confined are known as the prisoner set. An initial point z_0 chosen from the prisoner set generates a sequence that remains in a numerical prison no matter how many generations of the sequence are computed. The shape of the prison depends on the value of c that is chosen. For a point z_0 outside the prisoner set, the sequence z_k moves away from the center of the plane and escapes toward infinity. The prisoner set and the escape set are separated by an infinitely narrow boundary known as the Julia set [*see illustration above*].

Amazingly, the Julia set can also be obtained using the multiple-reduction copying machine by refitting it with special lenses that reverse the action of $g(z)$. The inversion of $g(z) = z^2 + c$ consists of two transformation functions, $f_1(u) = +(u - c)^{1/2}$ and $f_2(u) = -(u - c)^{1/2}$. (In these functions, c is the familiar control parameter, and u is the chosen input value.) These two functions can be considered the "reductions" performed by the copying machine. Repeated operations of the machine then cause randomly chosen points to move toward the Julia set.

The presence of the square root in the equations means that the copying machine no longer applies a uniform reduction factor. Moreover, because this transformation is nonlinear, straight lines are now imaged onto curved lines. From an initial image two

Two lens systems graphically reverse the quadratic transformation that defines the Julia set. The lenses perform the transformations $+(z - c)^{1/2}$ and $-(z - c)^{1/2}$, the inverses of $z^2 + c$. The limiting image of the copying machine is a Julia set.

smaller images emerge, then four, then eight, until the limiting image gradually takes shape [*see bottom illustration on preceding two pages*]. As with linear fractals, the limiting image does not depend on the particular initial image but is completely determined by f_1 and f_2 or, equivalently, by the choice of parameter c.

Now comes one of the most difficult but fascinating problems of fractal geometry. Returning to the metaphor of language, the problem can be expressed as a question: What are the grammatical rules of the quadratic dialects? In mathematical terms the question is, Does an order principle underlie the infinite variety of Julia sets?

The search for the answer has led to one of the most beautiful discoveries of experimental mathematics. The solution lies in the fact, known to Julia and Fatou, that for every control parameter c the resulting fractal image falls into one of two categories. The Julia set may be a single connected piece, or it may consist of an infinite number of disconnected points, like dust.

Suppose one plots a dot for every control parameter c in the complex plane that belongs to a connected Julia set and leaves blank every c that belongs to a disconnected one. The result is the now famous Mandelbrot set, a fractal of truly remarkable richness.

Obviously, one needs to know if a given Julia set is connected in order to decide whether a point c belongs to the Mandelbrot set. One of the great successes of Julia and Fatou was their discovery that this difficult problem can be solved by a simple calculation. Consider the sequence of values of z_k produced by $g(z) = z^2 + c$ when the initial point z_0 equals zero. In this way, attention is focused on the crucial factor, the control parameter c. The resulting sequence is $0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$. If this sequence does not escape toward infinity, the associated Julia set is connected, and the point c belongs to the Mandelbrot set.

Each portion of the Mandelbrot set characterizes a family of related Julia sets. For example, the heart-shaped main body of the Mandelbrot set characterizes Julia sets that look like crumpled circles. Although the Mandelbrot set is not exactly self-similar as are the Sierpinski triangle and fractal fern, it has a related characteristic: magnifying the boundary of the Mandelbrot set reveals an endless number of tiny copies of the set. The wealth of shapes and structures in the Mandelbrot set can be appreciated only when one inspects them in such minute detail.

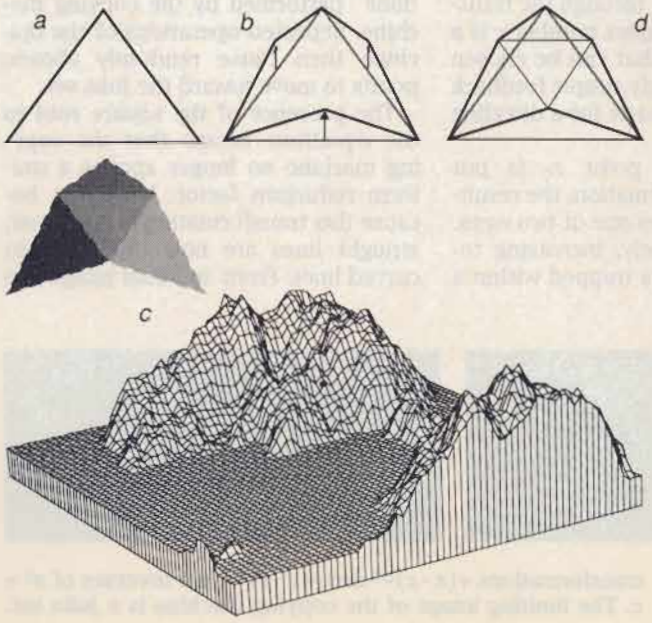
Perhaps the most fascinating aspect of the Mandelbrot set is that it functions as an endlessly efficient image storer: besides classifying Julia sets as connected or not connected, the Mandelbrot set also functions as a direct, graphic table of contents for an infinite number of Julia sets. Enlarging the Mandelbrot set around a point c on the edge of the set reveals forms that are also the building blocks of the Julia set associated with that point c. The mathematical rigor of this finding has not yet been settled, however. Tan Lei, a noteworthy young researcher now at the University of Lyon in France, has shown that the Mandelbrot set functions in this manner for most parameter values c that lie exactly on the boundary of the Mandelbrot set.

The properties of the Mandelbrot set have been and continue to be a great challenge to mathematical research. Enormous progress has been made through the fusion of mathematical theory and computer-graphics experiments, particularly in the fundamental work of Adrien Douady of the École Normale Supérieure in Paris and John H. Hubbard of Cornell University.

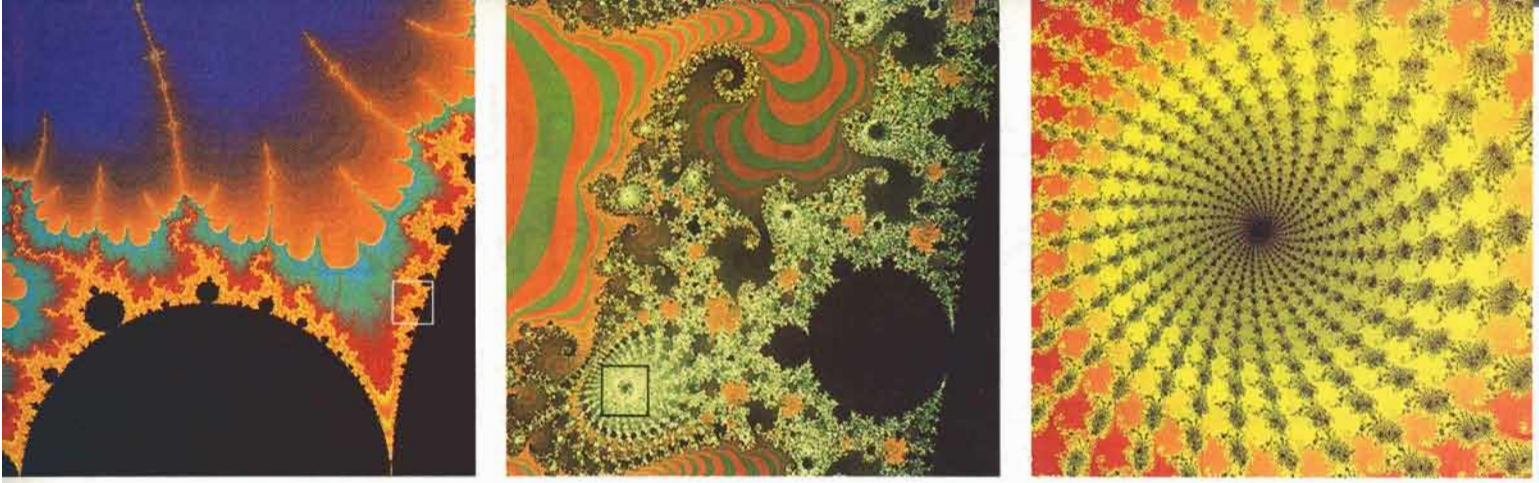
By far the most successful work in this area has been the so-called electrostatic potential of the Mandelbrot set. Imagine the Mandelbrot set being equipped with an electric charge. One could measure the potential by placing a point charge outside the set and measuring the electrostatic force on that point. It turns out that the computation of the potential is closely related to the series $0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$ that is used to determine whether or not a point c belongs to the Mandelbrot set.

Producing a three-dimensional representation of the potential has proved cumbersome, particularly in animations used to study the Mandelbrot set. Closer examination of the computer-graphics properties of the potential recently has made it possible to reduce the cost in computer time by an order of magnitude. As a result, researchers, including us, are increasingly exploring the Mandelbrot set by means of computer-animated video. Related work is also progressing on three-dimensional potential renditions of other fractals.

All fractals discussed up to this point can be considered deterministic. Although random processes (such as the roll of a die) may help generate fractal images, they do not have any impact on the final fractal form. The situation is completely dif-



FRACTAL LANDSCAPES can be created by random fractals using the midpoint-displacement method. The midpoints of the edges of a triangle (*a*) are connected by line segments and moved up or down, out of the image plane (*b*). This creates four smaller triangles on which the procedure is performed repeatedly. A distribution law assigns the amount of displacement and so defines the roughness of the fractal terrain. A computer-graphics program then fills the triangles with appropriate shading (*c*). The result can look startlingly real (*d*).



MANDELBROT SET reflects the order underlying the infinite variety of Julia sets. All points of the Mandelbrot set represent values of the parameter *c* that yield connected Julia sets. If the point *c* lies outside the Mandelbrot set, the associated Julia set is unconnected. The Mandelbrot set contains an unimaginable wealth of detail. Three zooms into the set reveal similar, repeating structures, including miniature copies of the overall Mandelbrot set, along with many new and different forms. If the entire set were shown at the scale of the image on the right, it would be the size of 100 football fields.

ferent for another class of fractals, the so-called random fractals.

One fractal of this type may begin with a triangle lying in an arbitrary plane. The midpoints of each side of the triangle are connected, breaking the triangle into four smaller triangles. Each midpoint is then shifted up or down by a randomly selected amount. The same process is applied to each of the smaller triangles, and the process is repeated ad infinitum. Over repeated iterations, an increasingly detailed surface begins to form.

In this midpoint-displacement method, the random amounts that the midpoints are raised or lowered are guided by a distribution law, which can be adjusted to obtain a close approximation of the surface to be modeled. For a model of a relatively smooth surface, the transformations should invoke a rule in which the amount of midpoint displacement becomes very small after only a few iterations. Such a rule adds only small bumps onto the overall contour. To represent a rough surface, such as the topography of a mountain range, it makes more sense to allow the displacement amounts to drop slowly with each iterative step.

This method of building surfaces has many applications. It has been used to model soil erosion and to analyze seismic patterns with a view to understanding changes in fault zones. Richard F. Voss, one of Mandelbrot's colleagues at IBM's research center, has employed the concept to create images of planets, moons, clouds and mountains that look remarkably real [*see illustration on opposite page*].

Regardless of their origin or method of construction, all fractals share one important feature: their roughness, complexity or convolutedness can be measured by a characteristic number, the fractal dimension. The various conceptual definitions of fractal dimension more or less date back to the 1919 work of Felix Hausdorff, a mathematician at the University of Bonn.

Following Mandelbrot's ideas, fractal dimension can be determined by a box-counting scheme. Imagine a complex shape that is masked with a lattice of squares plotted on graph paper. Some squares will contain part of the shape; others will be empty. The number N of nonempty squares depends on the given form and the mesh, or square size, E of the lattice. N is postulated to be proportional to $1/E^D$ (the finer the mesh, the more nonempty squares). The exponent D is the dimension. For a planar figure such as a circle, reducing the mesh by one half should multiply the number of nonempty squares by four (two squared) because the figure has a dimension of two. For a fractal, the number of nonempty squares would be multiplied by a slightly larger, or smaller, fractional value.

The above process is not restricted to mathematical objects or forms contained within a plane. One can also calculate the fractal dimension for real entities such as rivers, clouds, coastlines, trees, arteries, or villi of intestinal walls. Human arteries, for example, have a fractal dimension of about 2.7.

In addition to its usefulness for describing the complexities of natural objects, fractal geometry offers a welcome opportunity for the revitalization of mathematics education. The concepts of fractal geometry are visual and intuitive. The forms involved have a great aesthetic appeal and a wide variety of applications. Fractal geometry therefore may help to counter the perception that mathematics is dry and inaccessible and may motivate students to learn about this puzzling and exciting realm of study.

Scientists and mathematicians themselves have experienced a childlike wonder at the new and rapidly evolving language of fractals. As Mandelbrot himself wrote:

"Scientists will...be surprised and delighted to find that not a few shapes they had to call *grainy, hydralike, in between, pimply, pocky, ramified, seaweedy, strange, tangled, tortuous, wiggly, wispy, wrinkled*, and the like, can henceforth be approached in rigorous and vigorous quantitative fashion.

"Mathematicians will...be surprised and delighted to find that [fractal] sets thus far reputed exceptional...should in a sense be the rule, that constructions deemed pathological should evolve naturally from very concrete problems, and that the study of Nature should help solve old problems and yield so many new ones."

FURTHER READING

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