The Mandelbrot Set:  
A Paradigm for Experimental Mathematics

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Since the discovery of the Mandelbrot set in 1980, the sciences have been enriched by one of the most complicated and beautiful objects which they have ever seen. Its complexity and richness of structures and forms and the depth of mathematical problems connected with these is in amazing contrast to the simplicity of its generating code and needed the power of modern computer graphics to be revealed. In fact, a typical code needs only a few lines based on the simple dynamical system

\[ x_{n+1} = (x_n)^2 + c, \quad n=0,1,2,\ldots \]

(1) \[ x_0 \in \mathbb{C} \]

in the complex plane \( \mathbb{C} \). It exemplifies in a most beautiful way the power of experimental and computational methods in higher education. The Mandelbrot set, figure 1, embodies several important paradigmatic characterizations for the complex dynamical behavior of (1). For example, it is the bifurcation set of periodic orbits of (1) and describes various routes from order into chaos. Our interest is in its interpretations for a morphology of fractal basin boundaries, known as Julia sets. For example, values of \( c \) on the boundary of the Mandelbrot set locate a crisis in the associated Julia set of (1). The crisis is that for \( c \)-values from the Mandelbrot the associated Julia sets are connected, while for \( c \)-values outside the Mandelbrot set they are totally disconnected (a cloud of infinitely many points).
Figure 1. The Mandelbrot Set

Formally one defines the Julia set for $f_c(x) = x^2 + c$ as follows: Fix $c \in \mathbb{C}$. Then let $K_c$ be the set ($x_0 \in \mathbb{C}$; $x_n$ remains bounded for all $n$), where $x_n$ is obtained by (1). Since all finite periodic points are in $K_c$ one has that $K_c \neq \emptyset$. The complement of $K_c$ in $\Sigma = \mathbb{C} \cup \{\infty\}$ is denoted by $\Lambda_c(\infty)$ and can be interpreted as the basin of attraction of the attractive fixed point $\infty$ of $f_c$, i.e. $\Lambda_c(\infty) = \{x_0 \in \mathbb{C}; x_n \to \infty \text{ with } n \to \infty\}$. Now the Julia set of $f_c$ is defined as $J_c = \partial K_c = \partial \Lambda_c(\infty)$.

It is known from the classical work of G. Julia and P. Fatou that the $J_c$'s obey a simple dichotomy: $J_c$ is

- either connected,
- or totally disconnected.
Figure 2 shows a collection of a few Julia sets of both types. For example, figure 2e is obtained for \( c = i \); figures 2d and 2f are disconnected and all other Julia sets in figure 2 are connected. This is, however, a rather incomplete list; in fact, there are infinitely many significantly different Julia sets, namely one for each choice of \( c \) in (1). The Mandelbrot set

\[
M = \{ c \in \mathbb{C} : J_c \text{ is connected} \}
\]

can be seen as an order-principle in that infinite variety of \( J_c \)'s. It is not hard to show that the dynamical behavior of the critical point \( c \) determines \( M \) numerically

\[
M = \{ c \in \mathbb{C} : f_c^k(0) \text{ remains bounded for all } k \}.
\]

Here \( f_c^k \) denotes the \( k \)-th iterate of \( f_c \), i.e., \( f_c^k = f_c (f_c^{k-1}) \).

![Figure 2. Collection of Julia Sets](image-url)
On of the most striking interpretations of the Mandelbrot set is that it can be viewed as a "one picture dictionary" of these infinitely many shapes of Julia sets. This gives some flavor of its almost unimaginable complexity which still would be in the dark without the advent of modern computer graphics. This note is devoted to an intuitive discussion of this amazing "image compression".

Our first experiment in figure 3 illustrates a beautiful result of Tan-Lei [4]. A region along the cardioid of M (see (a) in figure 3) is continuously blown up and stretched out, so that the respective segment of the cardioid becomes a straight line segment. The result is shown in (b). The effect of this particular blow up is that now each member of a sequence of certain satellites (colored solid black) has the same size.\(^1\) Attached to the satellites one recognizes dendrite structures with 3, 4, \ldots, 10 major branches (from top to bottom). Choosing as \(c\)-values the major branch points of these structures, which are examples of Misiurewicz points, (see for example arrow in the top dendrite) one obtains 8 Julia sets which are shown on the right of (b). Remarkably, our blow up of the Mandelbrot set reflects certain structural and combinatorial aspects of these Julia sets in an amazing correspondence, which is not accidental. In fact this phantastic property of the Mandelbrot set is true everywhere along its boundary. More precisely, it is true for a dense subset of the boundary of M - the set of Misiurewicz points.

A point \(c\) is called a Misiurewicz point provided it is preperiodic, i.e. \(f_c^k (c) = c^*, \ k > 0, \) and \(f_c^m (c^*) = c^*\) for some \(m > 0.\) In other words, \(c\) is a Misiurewicz point provided the orbit of \(c\) under \(f_c\) is eventually periodic but not periodic. If \(m\) is the minimal period of \(c^*\) the number \(\rho = (f_c^m)'(c^*)\) is called the eigenvalue of the \(m\)-cycle \(c^*.\) It is known that \(|\rho| > 1.\) Now Tan-Lei's result is:
Figure 3.
THEOREM ([4])

Let $c$ be a Misiurewicz point and $\rho$ its eigenvalue. Then there is $\lambda \in C \setminus \{0\}$ and a closed subset $Z$ in $C$ such that

1. $\rho Z = Z$
2. $\rho^n(t_c(K_c)) \to Z$
3. $\rho^n(t_{-c}(M)) \to \lambda Z$.

($t_c$ denotes the translation by $-c$ and $\to$ is convergence in a suitably modified Hausdorff metric.)

The theorem can be interpreted in the following way: Suppose one magnifies the Mandelbrot set around a Misiurewicz point $c$ and compares with a magnification around the point $c$ in $K_c$ by the same blow up factor. Then in a small window the magnifications will be essentially indistinguishable, provided the blow up factor is large. But there is more and this will be discussed in our next experiment. For that let $c = 1$. Then $f_1(i) = -1+i$, $f_1(-1+i) = -i$ and $f_1(-i) = -1+i$. i.e. $c = i$ is a Misiurewicz point. Moreover $\rho = (f_1^2)'(-i) = \sqrt{2} \exp(2\pi i/8)$. Figure 4 shows 12 blow ups of $M$ around $c = i$. The blow up factor from one to the next is $\sqrt[3]{2}$. We observe an almost perfect selfsimilarity as we go from a to $1$ (compare figure 4d with 41). Moreover the tip of the dendrite at $c = i$ just appears to be rotated by $45^\circ$ in the subsequent blow ups from d to l, so that after a blow by $32^4$ the tip of the dendrite just appears as if it had been rotated by $360^\circ$. In other words, the tip of the dendrite is actually a spiral, as it should be according to $\rho Z = Z$ in Tan-Lei's result. This fact comes as a surprise for those who believed to be somewhat acquainted with $M$ based on computergraphical studies.
Figure 4. Blow ups of \( M \) around \( c = i \).
The selfsimilarity seen at Misiurewicz points is somehow misleading in regard to the complexity which one finds if one zooms in onto the Mandelbrot set. Figure 5 shows a sequence of 7 successive magnifications at the boundary of \( M \). The window for each blow up is indicated by a frame. Though images of \( M \) and its blow ups could be obtained from the numerical characterization (3) in principle, this method is usually not a suitable graphical method (see [2,3] for a detailed discussion of graphical methods for complex dynamical systems). Our experiments in figure 5 use a method which estimates the euclidean distance \( d(c, M) \) of a point \( c \) outside of \( M \) to the boundary of \( M \). This method is based on an estimate by J.Milnor which claims that

\[
(4) \quad d(c, M) \leq 2 \sinh(G(c))/|G'(c)|,
\]

where \( G:C \setminus M \to \mathbb{R}_+ \) is the potential of \( M \). Using this and the representation

\[
(5) \quad G(c) = \lim_{k \to \infty} 2^{-k} \log(|z^k_c(0)|)
\]

one derives an efficient code for the graphical representation of the Mandelbrot set (see [3])².

Our final experiment (in color) discusses the similarity between the Mandelbrot set and Julia sets for points in the interior of \( M \). Figure 5f shows a detail of \( M \), the center of which reveals a miniature copy of a Mandelbrot set. Choosing \( c \) from the center of this miniature copy the dynamical system (1) has an attractive periodic cycle of order 45. Figure 6(color) shows a global view of the Mandelbrot set and color figure 7(color) shows the blow up, which is identified by the yellow dot in color plate figure 6. The magnification here is identical with figure 5f though the method of its generation is quite different³.
Figure 5. Successive Blow Ups at the Boundary of Mandelbrot Set
Figure 5. (continued)
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Figure 5. (continued)
Figures 8 and 9 (color) show the Julia set and a blow up corresponding to the c-value from the center of figure 7 (color). The yellow dot in figure 8 (color) identifies the magnification shown in figure 9 (color). The dot marks a window centered at c. The magnification factor for 4 is the same as for figure 7 (color). However, the structure in figure 9 (color) is shown after a rotation of approximately 55°. The similarity of figures 7 and 9 (color) is striking and is subject of current research.

The true complexity of these experiments can only be revealed using color graphics, as for example in [2].

1) The satellites correspond to stability regions of certain periodic cycles of (1), i.e., for c values from a particular satellite the dynamical system (1) admits an attractive cycle of some fixed order: from top to bottom of figure 3 (II) these are stable 3-, 4-, ..., 10-cycles. The magnification factor is based on a conjecture, which says that the areas of the satellites along the cardioid decay like 1/p where p is the period identified by a satellite.

2) Distance Estimator Method for M: Choose \( N_{\text{max}} \) (maximal number of iterations) and \( R \) large, e.g., \( R = 100 \).

For each \( c \) one determines a label \( l(c) \) from \{0, 1, 2\} (0 for \( c \in M \), 1 for \( c \) close to \( M \), 2 for \( c \) not close to \( M \)).

Compute
\[
x_{k+1} = (x_k)^2 + c, \quad x_0 = 0, \quad k = 0, 1, 2, ...
\]
until either \( |x_{k+1}| \geq R \) (then \( l(c) = 2 \))
or \( k = N_{\text{max}} \) (then \( l(c) = 0 \)).

If \( l(c) = 2 \), then \( c \) is still a candidate for a point close to \( M \).
Thus, we try to estimate its distance, having saved the orbit \( \{x_0, x_1, ..., x_n\} \), where \( n \) is the first index such that \( |x_n| \geq R \):

\[
x_{k+1}' = 2x_kx_n + 1, \quad x_0' = 0, \quad k = 0, 1, ..., n
\]

If \( |x_{k+1}'| = \text{OVERFLOW} \), then \( l(c) = 1 \).

For in the course of the iteration (6) we get an overflow, then \( c \) should be very close to \( M \), thus we label \( c \) by 1. Finally, estimate the distance of \( c \) from \( M \):

\[
\text{if } \left| \frac{x_n}{x_{n-1}} \right| < \text{DELTAL}, \quad \text{then } l(c) = 1
\]

otherwise \( l(c) = 2 \).

It turns out that the images depend very sensitively on the various choices (\( R, N_{\text{max}}, \text{OVERFLOW}, \text{DELTAL} \) and blow-up factor when magnifying \( M \)). Therefore, it is appropriate to extend the labeling function by estimating points which have a distance given by a fraction of \( \text{DELTAL} \) (\( r \text{DELTAL} / a, r = 1, ..., 5 \)).
Using a proper color saturation for each of these extended labels usually gives very satisfactory results.

3) Level-Set Method for \( \mathbb{M} \): Fix a square lattice of pixels in the \( \mathbb{C} \)-plane, choose a large integer \( N_{\text{max}} \) (iteration resolution) and an arbitrary set \( \Lambda \) (target set) containing \( \infty \). For example, \( \Lambda = \{ z : |z| > 1/3 \} \). Now we assign for each pixel \( p \) from the lattice an integer label \( L(p;\Lambda) \) in the following way: \( p \) identifies a number \( c \), e.g., the center of \( p \)

\[
L(p;\Lambda) = \begin{cases} 
  k, & \text{provided } f_c^k(0) \in \Lambda \text{ and } 1 \leq k \leq N_{\text{max}} \text{ and } f_c^j(0) \notin \Lambda \text{ for } 0 \leq j < k. \\
  \emptyset, & \text{else.}
\end{cases}
\]

The interpretation of \( L(p;\Lambda) \geq 1 \) is obvious: \( p \) (or rather the value \( c = f_c(0) \) which \( p \) represents) escapes to \( \infty \) under the iteration of \( f_c \) and \( L(p;\Lambda) \) is the "escape time" - measured in the number of iterations - needed to hit the target set around \( \infty \). The collection of points of a fixed label, say \( k \), constitutes a level set, the boundary of which is the union of two circle-like curves, provided \( \Lambda \) is the complement of a large disc.

References


