A 4-Parameter Critical Power Model for Optimal Pacing Strategies in Road Cycling

Thorsten Dahmen

University of Konstanz, thorsten.dahmen@uni-konstanz.de

Introduction

Minimum-time pacing strategies in road cycling may be computed as solutions of optimal control problems. Both a mechanical model for road cycling power and speed and a physiological endurance model, that quantifies the energy resources and power limits of an athlete, are involved.

This contribution introduces a 4-parameter critical power model that arises from a combination of the 3-parameter critical power model (3PCPM), [4], and an exertion model developed by Gordon [2]. It has the form of a constrained dynamical system and is designed for the computation of minimum-time pacing strategies.

Previous work

In [1], we computed minimum-time pacing strategies using the 3PCPM, [4], and an exertion model, [2], for synthetic continuously varying slope profiles.

The 3PCPM extends the classical critical power model, that involves critical power P_{c} and anaerobic work capacity E_{a} by a parameter for maximum power at rest P_{max}. It can be illustrated as an hydraulic system as depicted in Figure 1a. For a pedalling power demand $P > P_{\mathsf{c}}$, the mechanical work beyond critical power is discounted from the remaining anaerobic resources

$$\dot{e}_{\mathsf{a}} = P_{\mathsf{c}} - P \ , \tag{1}$$

which are initially filled with E_{a} when the athlete is at rest.

The maximum power P_{m} is defined by a function decreasing linearly from P_{max} at rest to critical power in the exhausted state when the anaerobic resources are is empty ($e_{\mathsf{a}} = 0$) :

$$P\left(t\right) \leq P_{\mathsf{m}}(e_{\mathsf{a}}) = P_{\mathsf{c}} + \frac{P_{\mathsf{max}} - P_{\mathsf{c}}}{E_{\mathsf{a}}} e_{\mathsf{a}} \ . \tag{2}$$

Thus, for constant power exercises with $P\left(t\right) = \bar{P}$, exhaustion occurs at time T, when P_{m} falls below the power demand $\bar{P}$.

In contrast, the exertion model abandons the power constraint (2) and defines an exertion rate that is inversely proportional to time to exhaustion for arbitrarily varying power:

$$\dot{e}_{\mathsf{ex}}\left(P\right) = -\frac{E_{\mathsf{a}}}{T} = \frac{(P_{\mathsf{max}} - P_{\mathsf{c}})(P_{\mathsf{c}} - P)}{P_{\mathsf{max}} - P} \ , \tag{3}$$

thus implicitly limiting power by the infinitely growing exertion rate for P approaching P_{max}.

Methods

We develop our 4-parameter exertion model by combining the advantageous properties of both models.

From a physiological viewpoint maximum power is constrained depending on the duration of an exercise because either ATP utilization itself or its replenishment of secondary energy resources is rate limited [3]. It is impossible to model these complex details quantitatively but a single separate energy rate limit such as the power constraint (2) of the 3PCPM can be handled and is clearly favourable compared to the further simplification in Gordon's exertion model.

Moreover, using the principle of optimality, it can be shown that with the constraint (2) a minimum-time pacing strategies will always feature a final spurt at the end. In contrast, when using Gordon's exertion model, the athlete is necessarily exhausted on the last section where his power is limited to critical power.

However, we prefer the non-linear dependence of the exertion rate in (3) to the linear dependence in the 3PCPM (1). With high intensity power, glycolysis is increasingly impeded for example by the accumulation of lactic acid. For a load beyond the anaerobic threshold, the lactate concentration grows disproportionately high and leads to a reduced efficiency of power production that our heavily simplifying model may account for by an exertion rate that grows non-linearly. It is not realistic that for an athlete a constant power represents the same load as any time variant power with the same average. Furthermore, concerning pacing strategies, the linearity leads to a singular control problem that can cause numerical difficulties.

This is not the case for minimum-time strategies that rely on Gordon's exertion model. However, due to the pole at $P = P_{\mathsf{max}}$ high intensity power is heavily punished, leading to very little variations in the optimal pedalling power.

Therefore, we seek a modification of (3) that takes a finite value at $P = P_{\mathsf{max}}$ while roughly preserving the characteristics for lower power. We may imagine the pole in the graph at a very high hypothetical pedalling power, that in practice can never be achieved due to the maximum power constraint that we adopted from the 3PCPM. The definition of an exertion rate

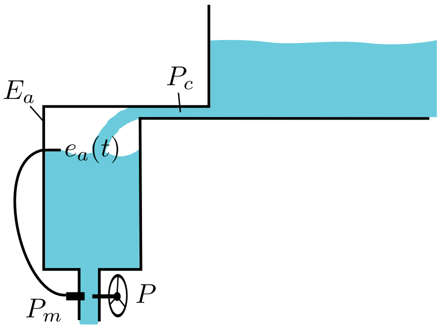
$$\dot{r} = \alpha \frac{(P_{\mathsf{max}} - P_{\mathsf{c}})(P_{\mathsf{c}} - P)}{\alpha P_{\mathsf{max}} - P} \tag{4}$$

with $\alpha \geq 1$ achieves these requirements and adds a fourth steering parameter α to the model. The graph of (4) is plotted in Figure 3.

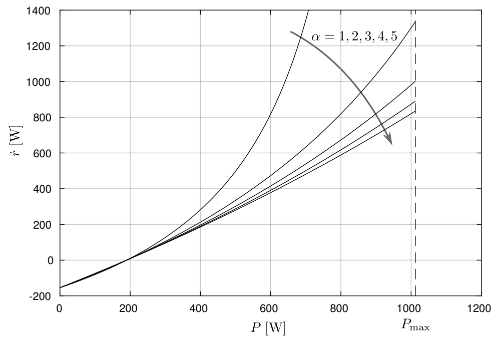
Results

Figure 2 illustrates a minimum-time pacing strategy for a typical setup on a track near Pfyn in Switzerland. It was computed with the MATLAB optimal control package GPOPS-II, [5].

Initially, high power is indicated to accelerate from rest. Thereafter, both power and speed vary according to the slope profile $\frac{\mathrm{d}h}{\mathrm{d}x}$ that contains many details. Differential gps was used to measured the height in order to ensure the required accuracy. At the end of the steepest section around 4000 m, the athete uses his maximum power to reach the top as soon as possible, where he is completely exhausted. On the following descent he recovers partially before he enters the final spurt on the last 480 m.



(a) The 3PCPM depicted as a hydraulic system. The black line connects a sensor for the level of anaerobic resources e_a to the size of the outlet according to the power constraint (2).



(b) Graph of the exertion rate (4) of the 4-parameter exertion model. For $\alpha = 1$ the rate is identical to Gordon's exertion rate defined in (3).

Figure 1: Characteristics of the physiological models.

Conclusions

The components of the 3PCPM and Gordon's exertion model may be combined to a 4-parameter exertion model. The exertion rate grows non-linearly with power load but not excessively for high intensity power. A separate power constraint limits the maximum available power. Solutions to minimum-time pacing strategies are numerically stable for complex realistic slope profiles, reveal a balanced variability in power and speed, and guarantee a final spurt.

References

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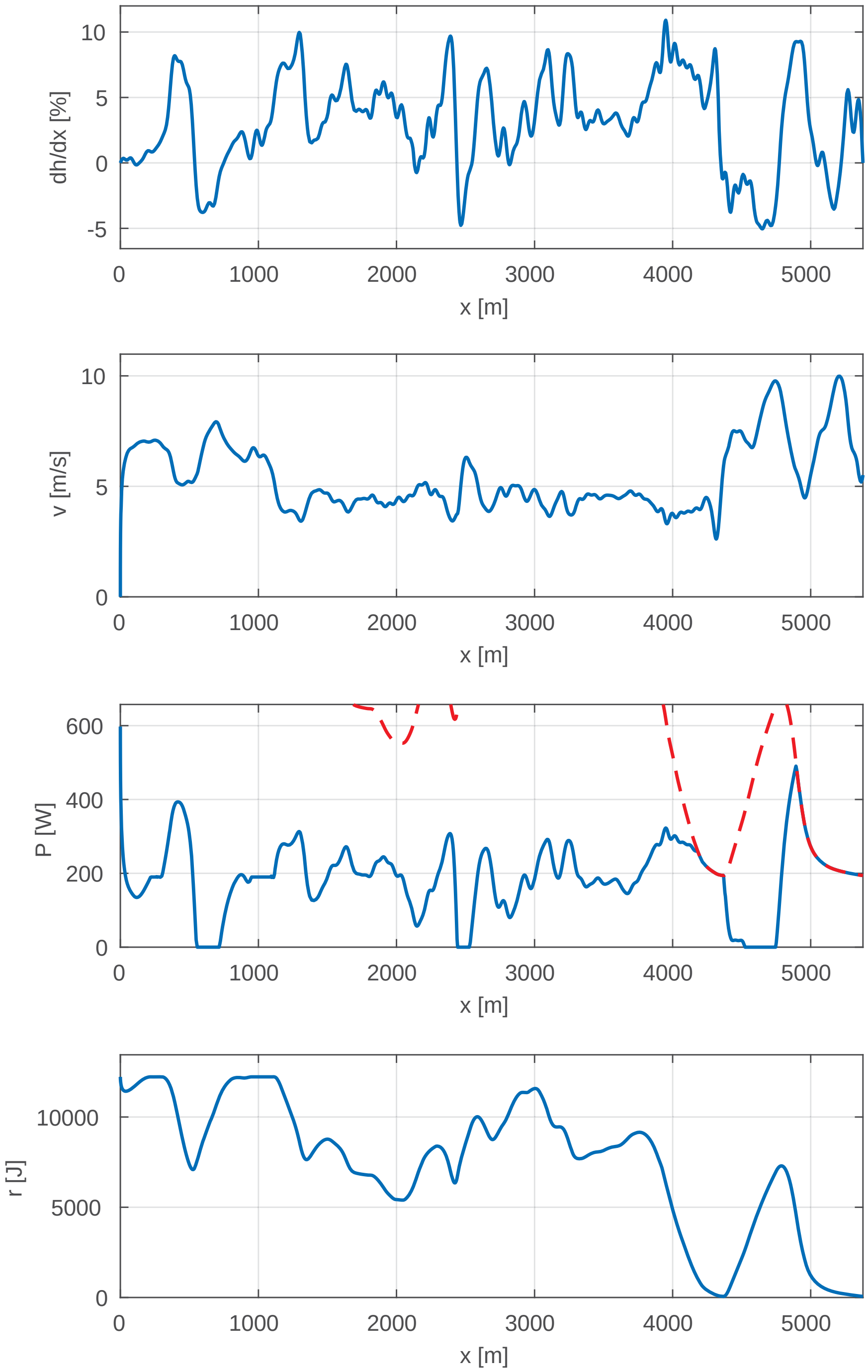

Figure 2: Minimum-time pacing strategy for the 4-parameter exertion model with $\alpha = 3$ and the real slope profile of the cycling track in Pfyn. The minimum time is $t_{\mathrm{f}}^{\star} = 18$ min 17.8 s. The red dashed line represents the maximum power P_{m}.