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# Optimization of Pacing Strategies for Cycling Time Trials Using a Smooth 6-Parameter Endurance Model

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**Abstract.** Computing the optimal pacing strategy for cycling time trials can be formulated as an optimal control problem, where a mechanical model and a physiological endurance model form the dynamical system and time to complete the track is to be minimized. We review approaches that use the 3-parameter critical power model to compute optimal pacing strategies and modify it to become a smooth 6-parameter endurance model. Due to its 3 additional parameters, it is more flexible to model the physiological dynamics appropriately. Besides, we demonstrate that this model has favourable numerical properties that allow to eliminate purely mathematical workarounds to compute an approximate optimal pacing for the original 3-parameter critical power model.

An established simplification of the 3-parameter critical power model is considered for a comparison of numerically computed optimal pacing strategies on an artificial track with continuously varying slope subject to these variants of the 3-parameter critical power model. It is shown, that the optimal pedalling power subject to the original model exhibits unrealistically large variations, which are smoothed heavily by the simplified model. The 6-parameter endurance model turns out to be a flexible model, that exhibits intermediate variations in the optimal pedalling power, while being numerically well behaved.

The methods used in this contribution are extensible and can be used for the computation of optimal pacing strategies in conjunction with more sophisticated physiological models.

**Keywords:** road cycling, physiological endurance model, numerical minimum-time pacing strategy

## 1. Introduction and related work

We are concerned with the optimal control problem, that seeks the optimal pacing, i.e. pedalling power $P$ and cycling speed $v$, of a cyclist in order to complete a track of length $L$ in minimum time $T$:

$$T = \int_0^L \frac{1}{v}\,\mathrm{d}x \;\rightarrow\; \min. \tag{1}$$

The dynamics of the mechanical model and two versions of an established physiological endurance model are described in the following two subsections.

### 1.1. The mechanical $P$-$v$-model

As established by [1], the relationship between pedalling power $P$ (control variable) and cycling speed $v$ (state variable) can be expressed as an equilibrium of the propulsive force $P/v$, discounted by the chain efficiency factor $\eta$, and resistance forces $F_{res}$ at the contact area between wheel tire and road. The resistance forces comprise forces due to gravity $F_g$, rolling resistance $F_r$, aerodynamic drag $F_a$, frictional losses in wheel bearings $F_b$, and inertia $F_i$. Substituting models for each resistance force and eliminating time yields the nonlinear ordinary differential equation

$$\frac{\eta P(x)}{v} = F_{res} = \underbrace{mgh'(x)}_{F_g} + \underbrace{mg\mu}_{F_r} + \underbrace{Dv^2}_{F_a} + \underbrace{\beta_0 + \beta_1 v}_{F_b} + \underbrace{\left(m + \frac{I_w}{r_w^2}\right) v'(x) v}_{F_i}\,. \tag{2}$$

The distance $x \in [0; L]$ is the independent variable. The other parameters are given in Tab. 1.

| | | | | | |
|---|---|---|---|---|---|
| $\eta$ | chain efficiency | $\mu$ | rolling resistance factor | $\beta_0$ | bearing friction factor |
| $m$ | mass (cyclist and bike) | $h(x)$ | height profile | $\beta_1$ | bearing friction factor |
| $g$ | gravity factor | $D$ | drag coefficient | $L$ | track length |

Tab. 1: Parameters of the mechanical bicycling model (2).

The initial value is defined as $v(0) = v_0$.

### 1.2. The physiological endurance model

Naturally, seeking the minimum time pacing strategy requires limiting the energy resources and maximum available power of the cyclist, which calls for a physiological endurance model. Hydraulic models, such as the 3-parameter-critical-power model, [2], can be used to extend the dynamical system, [3]. A new state for the remaining anaerobic energy resources $0 \leq e_a \leq E_a$ is introduced. It decreases from fully charged anaerobic resources $e_a(0) = E_a$ subject to the dynamics

$$e_a' = (P_c - P)/v \,, \tag{3}$$

where $P_c$ represents the critical power, which is the maximum power that the cyclist can hold for a very long time. Additionally, the maximum achievable power, $P_m(e_a)$ drops linearly with decreasing anaerobic resources form the total maximum power $P_m(E_a) = P_{max}$ to the critical power $P_m(0) = P_c$:

$$0 \leq P \leq P_m(e_a) = P_c + (P_{max} - P_c)\, e_a/E_a \,. \tag{4}$$

In the following, we refer to this physiological model as the model $A$.

For minimum-time *running*, a numerical approach, however only for flat tracks and without constraint on the runner's propulsive force, was recently presented by [6].

For the physiological model in [4], here referred to as model $B$, equations (3) and (4) are simplified: Time to exhaustion $T_{ex}(\bar{P})$ is calculated as the time for which a cyclist can hold a constant power $\bar{P}$ until the maximum achievable power $P_m(e_a)$ falls below $\bar{P}$. Arbitrarily allowing for time-varying power $P$, the reciprocal of this function scaled by $E_a$ is defined as the (time-related) exertion rate. Hence,

$$e_{ex}' = \frac{E_a}{v T_{ex}(P)} = \frac{P_{max} - P_c}{v} \cdot \frac{P_c - P}{P_{max} - P} \tag{5}$$

applies, where the initial exertion is $e_{ex}(0) = E_a$. Evidently, total exertion at the end of the track, i.e. $e_{ex}(L) = 0$ J is necessary for a pacing to be optimal. Thus, a power constraint such as (4) is avoided. For the optimization of pacing strategies, in [4], the inertia was neglected and only an analytic solution for synthetic and piecewise constant slopes was given. Eventually, [5] repealed these limitations and provided a numerical algorithm to compute an optimal pacing strategy on tracks of varying slope.

## 2. The smooth 6-parameter endurance model

In this contribution, we strive to revoke the simplification (5), and extend the 3-parameter-critical power model to a more realistic 6-parameter endurance model $C$. Concurrently, we show that this model features favourable properties regarding numerical algorithms to compute optimal pacing strategies for tracks of varying slopes.

### 2.1. Limitations of the 3-parameter critical power model *A*

One major shortcoming of the model $A$ is the linear relationship between the differential anaerobic resources $e_a'$ and the difference of critical and pedalling power in (3). A smaller recovery capability and a disproportionate decrease of $e_a'$ for higher pedalling power would certainly be more realistic.

A further problem, that concerns the mathematical treatment of the optimal control problem, originates in the fact that the control variable $P$ enters linearly in the system equations (2) and (3). Generally, this property induces that Pontryagin's Minimum Principle gives only incomplete information on the optimal control candidates, yet solutions of higher order conditions yield additional singular control candidates. For the problem at hand, one obtains three candidates for the optimal pedalling power $P^*$,

$$P^* = \begin{cases} 0 \\ P_m \\ P_{end} = F_{res} v/\eta \end{cases}, \tag{6}$$

where the first two constrained power candidates result from the Minimum Principle $P^* = \arg\min_P H$, where $H$ is the Hamiltonian of the system. The third candidate is a singular solution, which represents a moderate and steady endurance pacing $P_{end}$, [3]. These candidate solutions need to be pieced together to obtain the optimal pedalling power strategy $P^*$. For a sufficiently long track with moderate variations in the slope, one can guess, that the optimal sequence of power candidates is $P_m$ ; $P_{end}$ ; $P_m$. This sequence reflects the ideas, that a high initial power is necessary to gather momentum; the power cannot be maximal for a longer time since the anaerobic resources must be conserved and a final spurt is essential for optimality.

However, transitions between constrained and singular arcs cause discontinuities in the pedalling power, which degrade the performance of numerical algorithms that approximate $P^*$ with smooth functions. Even worse, singular arcs feature that $\frac{\partial^2 H}{\partial P^2} \equiv 0$ and hence the corresponding matrix in numerical algorithms is only semi-definite, [7].

## 2.2. Avoiding discontinuities and singularities

In order to make a numerical solution possible, one may apply the method of saturation functions and system extension, [8], and solve the following approximate optimal control problem for the model $A$: One formulates an unconstrained optimal control problem with the transformed control variable $\tilde{P}$, where

$$P = \psi(\tilde{P}\,; 0,0,1,P_m)\,,$$

with $\psi(z\,; Z_-, Z_0, Z_0', Z_+)$ being a strictly monotonic smooth saturation function as given in Fig.1.

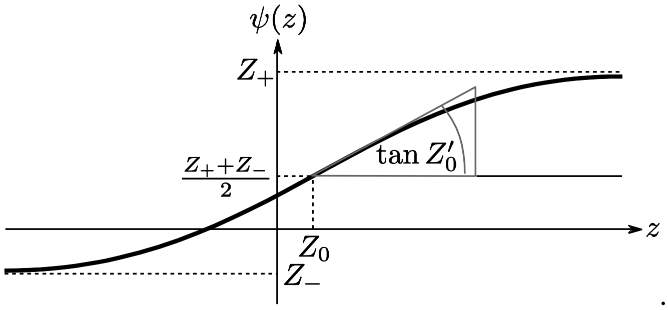


Fig. 1: Shape of the saturation function $\psi(z\,; Z_-, Z_0, Z_0', Z_+)$.

To obtain a finite $\tilde{P}$ as a solution, it is necessary to introduce a further regularization parameter $\epsilon$ into the performance criterion (1):

$$T = \int_0^L \frac{1}{v} + \frac{\epsilon}{2}\tilde{P}^2 \; dx \;\rightarrow \min. \tag{7}$$

Thus, large absolute values of $\tilde{P}$ are penalized, whereby a finite optimal $\tilde{P}^*$ is found. Note, that $P(\tilde{P}^*) \lesssim P_m$ on intervals, where the solution of the exact problem would be a constrained arc $P^* = P_m$. The solution of the approximate problem does not contain constrained arcs anymore. The control variable $\tilde{P}$ enters non-linearly into the system equations (2) and (3), prohibiting singular arcs. The optimal pedalling power $P^*(x)$ is a smooth function of the distance.

## 2.3. The 6-parameter endurance model

To make the model more realistic and to overcome the numerical issues simultaneously without the mathematical workaround in the previous subsection, we introduce the 6-parameter endurance model $C$: The decline of the anaerobic resource $e_a'$ in Equation (3) is replaced by

$$r_a' = \psi(P\,; p_0, P_s, s, 1)e_a'. \tag{8}$$

The parameter $p_0$ quantifies the recovery rate with vanishing pedalling power whereas $P_s$ and $s$ determine the shape of the disproportionate decrease of $r_a'$ with $P$. The boundary condition is unchanged: $r_a(0) = E_a$. Still, we impose a limit on $P$ as in Equation (4):

$$0 \le P \le P_m(r_a) = P_c + (P_{max} - P_c)\, r_a / E_a\,. \tag{9}$$

The differential anaerobic resource as a function of pedalling power for all three physiological endurance models is given in Fig. 2.

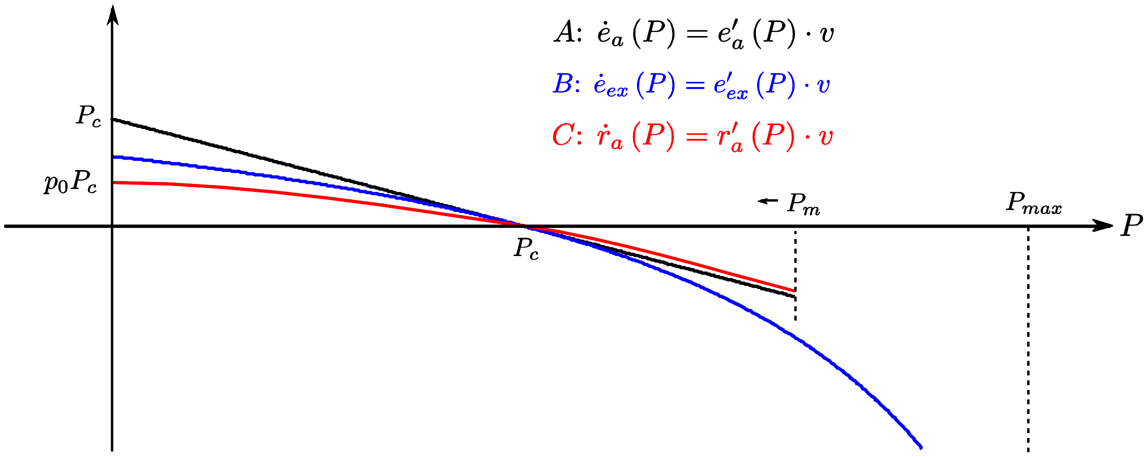


Fig. 2: Comparison of $\dot{e}_a$, $\dot{e}_{ex}$, and $\dot{r}_a$ as a function of $P$. With decreasing $e_a$ and $r_a$, respectively, $P_m$ decreases.

The additional parameters $p_0$, $P_s$, and $s$ can be determined together with $P_c$, $P_{max}$, and $E_a$ by curve fitting with physiological ergometer test, potentially including lactate and spiro-ergometric measurements. However, the parameter estimation is no matter of the contribution at hand, [9].

Clearly, due to the incorporation of three additional parameters, the human endurance may be modelled more precisely. Now, one of the system equations, (8), is a nonlinear function of the control variable $P$, whereby only non-singular control candidates can occur. The transitions between constrained and interior arcs involve a not necessarily smooth, but continuous optimal pedalling power.

## 3. Numerical Results and Discussion

For the following numerical example, we use the parameter values as given in Tab. 2 and the slope profile $h'(x)$ as depicted in Fig. 3a.

| | | | | |
|---|---|---|---|---|
| $\eta = 1$ | $\mu = 0.003$ | $\beta_1 = 0\ \frac{\text{Ns}}{\text{m}}$ | $P_{max} = 1234$ W | $p_0 = 0.5$ |
| $m = 84$ kg | $D = 0.17\ \frac{kg}{m}$ | $L = 2$ km | $P_c = 435$ W | $P_s = 500$ W |
| $g = 9.81\ \frac{\text{m}}{\text{s}^2}$ | $\beta_0 = 0$ N | $v_0 = 0.1\ \frac{\text{m}}{\text{s}}$ | $E_a = 12430$ J | $s = 0.005$ |

Tab. 2: Numerical values for the model parameters: the first four columns contain values adopted from [4], barring $v_0$ which is our arbitrary choice to avoid that the denominator of the LHS in (2) vanishes; column 5 contains reasonable but arbitrarily chosen values for the newly introduced parameters of the saturation function $\psi(z\,; Z_-, Z_0, Z'_0, Z_+) = \frac{Z_+ + Z_-}{2} + \frac{Z_+ - Z_-}{2}\tanh\big(Z'_0 \cdot (z - Z_0)\big)$, which was used in this example.

The artificial but continuously varying slope features a hills and a valley. From the optimal power $P^*$ in Fig. 3b of the *approximate* minimum-time problem with model $A$ (black) and regularization $\epsilon = 10^{-9}$ it is evident, that the optimal solution for the *exact* problem with model $A$ consists of the sequence $P_m$ ; $P_{end}$ ; $P_m$ ; $P_{end}$ ; $P_m$. For our model $C$ (red curve), the same sequence is observed. However the switching times are different and $P_{end}$ represents an interior non-singular arc. Model $B$ produces a smooth optimal pedalling power so that there is no sequence at all.

Quantitatively, model $A$ produces the largest variation in power, since the athlete does not spend disproportionately much energy at high power and benefits from large recovery rates at low power. Therefore, model $A$ is the only model that yields a significant acceleration in *advance* of an ascent and an optimal power well below critical power as the optimum on steep descents.

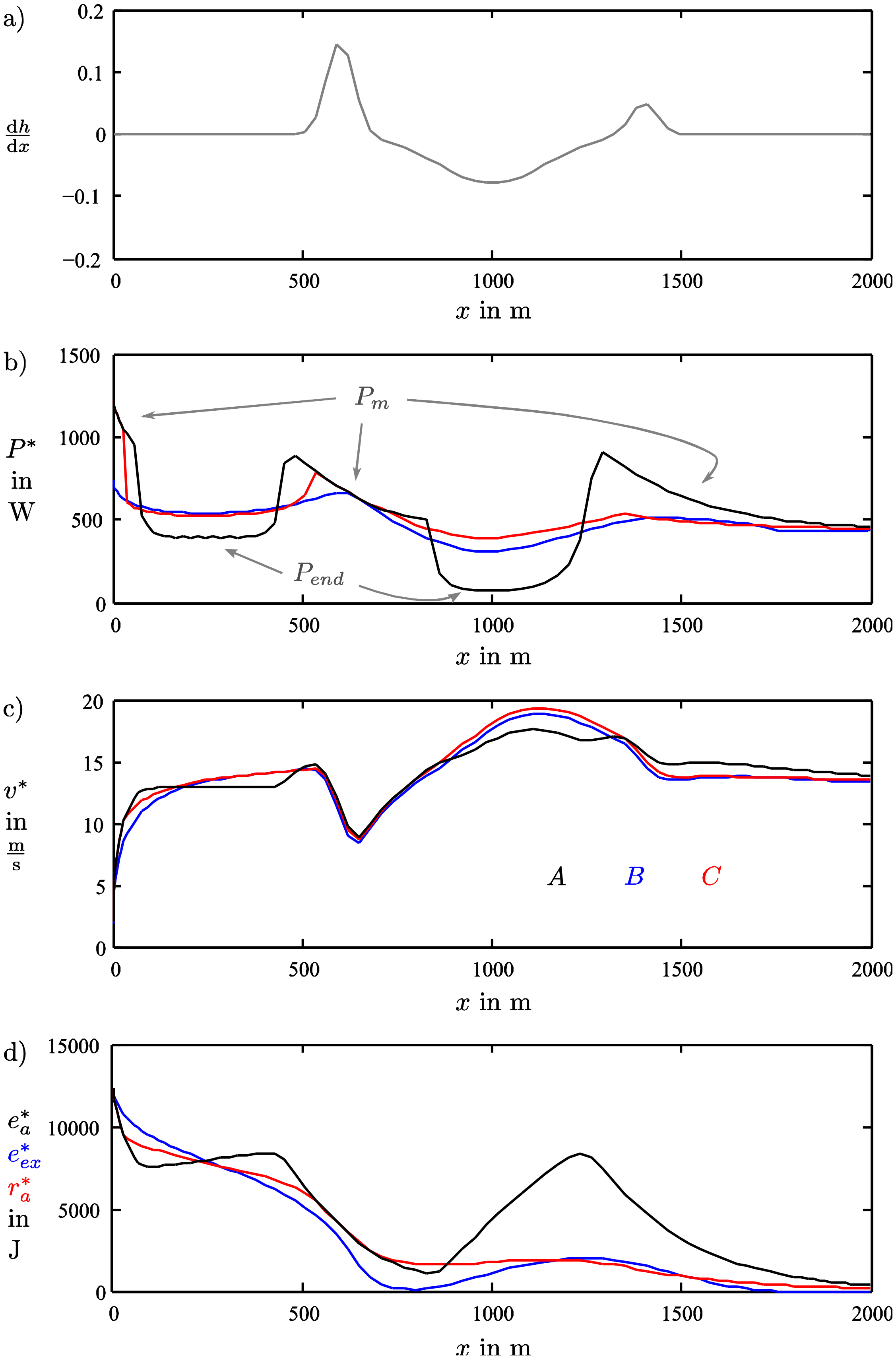


Fig. 3: (a) Synthetic slope profile of a track of length 2 km with first peak at 687 m distance and 12.8 m altitude, valley at 1326 m (-14.7 m altitude) and a final altitude of -10.6 m. ; (b-d) optimal pacing strategies using the approximate 3-parameter-critical-power model $A$ (black, $\epsilon = 10^{-9}$), the simplified model $B$ (blue), and the smoothed 6-parameter endurance model $C$ (red). The optimal solution for the *exact* problem with model $A$ consists of the sequence $P_m$ ; $P_{end}$ ; $P_m$ ; $P_{end}$ ; $P_m$. The optimal solution for the problem with model $C$ has the same sequence. The total times are 144 s for model $A$, 149 s for model $B$, and 148 s for model $C$.

Model $C$ yields moderate variations in the optimal power. The initial acceleration is very similar to model $A$, yet shorter. On the first hill, the power increase occurs later and less pronounced. Due to limited recovery in the valley, less energy is left for the final spurt than with model $A$, so that towards the end, the curves resemble those of model $B$. The total times to complete the course for all models are close between 144 and 149 s.

## 4. Conclusion

The smooth 6-parameter endurance model $C$ overcomes major limitations of the 3-parameter-critical-power model $A$ in terms of flexibility of recovery rates at low power and disproportionate increase of exertion at high power at the expense of additional parameters that have to be determined. It has favourable properties regarding numerical algorithms that compute the minimum-time pacing to complete a track of varying slope, whereas the solution with model $A$ requires a mathematical workaround. An alternative simplification, model $B$, is numerically well behaved, but compensates its lack for a maximum power constraint by disproportionately high exertion rates at high power. Thus, model $B$ is incapable of modelling high optimal power outputs at the beginning of a track or before and on steep ascents.

The numerical methods from [8] are generally applicable for constrained and singular optimal control problems. Thus, they could be used to develop numerical algorithms for minimum-time pacing strategies on tracks with varying slope, involving advanced physiological models. The generalized 3-component Morton-Margaria model, for which the optimal running strategy has been discussed qualitatively in [10], is an interesting candidate for such an analysis. Again, smoothing the dynamics to achieve more flexibility and numerical stability simultaneously could lead to a realistic and robust solution, provided that the question of parameter estimation, not considered in this contribution, was solved appropriately.

## 5. Acknowledgements

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