IMAGE CODING USING OVERLAPPING FRACTAL TRANSFORM IN THE WAVELET DOMAIN

B. Simon

Université catholique de Louvain
Laboratoire de Télécommunications et Télédétection
2 Place du Levant. 1348 Louvain-la-Neuve. Belgium
email: simon@tele.ucl.ac.be
phone: +32 10 47 80 71 fax: +32 10 47 20 89

ABSTRACT

Fractal and frequency decompositions lead to comparable image compression results. However, these two approaches are not incompatible and it is possible to express the operations performed by some fractal coders in the frequency domain. The aim of this paper is to propose an image representation technique combining both fractal and frequential decompositions thanks to the use of the wavelet multiresolution representation. This model takes into account a certain kind of self-similarity that can be translated in terms of simple operations on the high frequency coefficients of a subband decomposition of the image. The way to solve analytically the inverse problem is also described in this context.

1. INTRODUCTION

The image compression problem can be solved in two different ways. Either images are considered as a realization of a stationary random process or as a combination of spatial elements related to each other. In the first case, the image content can be decorrelated thanks to the use of frequency decompositions. This approach has lead to the well-known DCT or subband decompositions. In the second case, spatial models are used in order to describe the signal content. These tend to identify local and global correlation within the image and to express it in a compact form. However, as it is shown in this paper, these two solutions can be applied simultaneously in order to exploit the advantages of both.

Fractal coding methods are part to the second class of techniques because these take advantage of the self-similarity which is a spatial property of images. The self-similarity models identify the relationships between the frequency components of the signal at different locations and scales. It ensues that the inverse transformation can be done by following these links in an iterative way under some contractivity condition to obtain the decoded image[1][4]. The main fractal image compression algorithms [3] identify the self-similarity in the spatial domain. However, Barthel has proposed a method to combine the DCT and fractal image representations [2].

The merging between frequency and fractal decompositions formalisms can be conducted further for some fractal transformations. The relationships between the frequency components of the signal can be expressed very simply thanks to wavelet multiresolution representation of the signal. These elements have already been presented in previous articles for one-dimensional signals [6] [7].

In the present article, we apply and extend former results to images. The model that is described combines self-similarity and frequency coefficients in order to offer a perfect reconstruction scheme if these coefficients are not quantized. The potential applications of this model are related to image compression, texture synthesis or signal zooming.

2. 1D MODEL

The model developed in this paper is derived from the study of a self-similar model for one dimensional functions characterized by a spatial contraction factor of two and by a overlap between the self-similar components of the signal. This study has lead to some important conclusions that are synthesized hereafter. The functions are assumed to be square integrable.

In the present context, a one dimensional function $f(x)$ is self-similar if it can be expressed by

$$f(x) = \tau(\tilde{f}(x)) = \sum_l \alpha(l)\, \tilde{f}(2x - l) + r(x).$$

The function $\tilde{f}(x)$ contains the high frequency information of $f(x)$ and is defined more precisely in the next paragraph. The contractivity condition for $\tau(\cdot)$ is given by

$$\sum_l |\alpha(l)| < \sqrt{2}.$$

We shall refer to the first term of the transformation as the contractive term and to $r(x)$ as the independent term.

Given an multiresolution analysis characterized by its wavelet and scaling functions respectively denoted $\Psi(x)$ and

$\Phi(x)$, any function $f(x)$ can be decomposed onto the multiresolution basis in the following way

$$f(x) = \sum_n a_0^f(n)\, \Phi_{0,n}(x) + \sum_{j\geq 0} d_j^f(n)\, \Psi_{j,n}(x),$$

where $h_{j,n}(x) = \sqrt{2^j}\, h(2^j x - n)$. We define $\tilde{f}(x)$ as the sum of detail components of $f(x)$,

$$\tilde{f}(x) = \sum_{j\geq 0} d_j^f(n)\, \Psi_{j,n}(x).$$

The high frequency coefficients, $d_j^f(n)$, of a self-similar function $f(x)$ are linked to each other via

$$d_{j+1}^f(n) = \frac{1}{\sqrt{2}} \sum_k \alpha(k)\, d_j^f(n - 2^j k) + d_{j+1}^r(n)$$

This means that the contractive term of the self-similar model generates an inter-resolution interpolation process and the independent term adds a *non self-similar* correction to the estimate created by the interpolation. These correction coefficients are also called residues. They allow to control the loss of quality due to the self-similarity modelization.

3. 2D MODEL

The definition of self-similarity given for one dimensional signals can easily be extended in two dimensions. In this section, we define the concept of global and piecewise self-similarity and their implications on the wavelet multiresolution coefficients of two dimensional functions. This allows to design an image model combining self-similarity and frequency decompositions. An analytic way to estimate the parameters of the model is also given.

3.1. Global self-similarity

Given a wavelet multiresolution representation of $L^2(R^2)$ characterized by its scaling and wavelet functions, respectively denoted $\Phi(x;y)$ and $\Psi^1(x;y)$, $\Psi^2(x;y)$ and $\Psi^3(x;y)$, any function $f(x;y)$ can be decomposed onto an orthonormal basis given by

$$\{\Phi_{J,n_1,n_2}(x;y)\}_{n_1 . n_2 \in Z}$$

$$\text{and } \{\Psi^b_{j,n_1,n_2}(x;y)\}_{j,n_1,n_2 \in Z, j\geq J} \quad b = 1,2,3$$

where $h_{j,n_1,n_2}(x;y) = 2^j\, h(2^j x - n_1; 2^j y - n_2)$. It follows that any function $f(x;y)$ can be written as

$$f(x;y) = \sum_{n_1,n_2} a_J^f(n_1,n_2)\, \Phi_{J,n_1,n_2}(x;y)$$
$$+ \sum_b \sum_{j\geq J} \sum_{n_1,n_2} d_j^{f,b}(n_1,n_2)\, \Psi^b_{j,n_1,n_2}(x;y) \qquad (1)$$

We shall refer to the coefficient $a_J^f(n_1,n_2)$ as the approximation coefficients and to the $d_j^{f,b}(n_1,n_2)$ as the detail coefficients in the direction b at resolution 2^j.

If we denote

$$\tilde{f}^b(x;y) = \sum_{j\geq J} \sum_{n_1,n_2} d_j^{f,b}(n_1,n_2) \Psi^b_{j,n_1,n_2}(x;y),$$

the function $f(x;y)$ is said to be self-similar if each component $\tilde{f}^b(x;y)$ can be expressed as

$$\tilde{f}^b(x;y) = \tau^b\left(\tilde{f}^b(x;y)\right)$$
$$= \sum_{l_1,l_2} \alpha^b(l_1,l_2)\, \tilde{f}^b(2x - l_1; 2y - l_2) + \tilde{r}^b(x;y) \qquad (2)$$

The parameters $\alpha^b(l_1,l_2)$ are called the self-similarity parameters and are represented by matrix of size $L \times L$. Some remarks are to be done in relation to the previous definition of self-similarity:

- the model is chosen to be different along the three detail directions.
- more than two copies of the shrunken signal $\tilde{f}^b(2x;2y)$ can be added in order to compose the self-similar signal $\tilde{f}^b(x;y)$. This implies that the terms $\alpha^b(l_1,l_2)\tilde{f}^b(2x - l_1; 2y - l_2)$ will overlap in the spatial representation of the transformation.

3.2. Piecewise self-similarity

In the model presented so far, the function is composed of a single self-similar component. However, natural images do not exhibit this kind of self-similarity. A unique self-similar model cannot represent efficiently any image and must therefore be adapted to the local content of the signal. For natural images, the concept to be used is the piecewise self-similarity[4]. This means that local parts of the signal are similar to each other through a self-similarity relationship. A possible manner to express that uses the decomposition of the original signal into a sum of localized components denoted $f^b_{i_1 i_2}(x;y)$ such that

$$\tilde{f}^b(x;y) = \sum_{i_1,i_2} \tilde{f}^b_{i_1 i_2}(x - i_1; y - i_2). \qquad (3)$$

The local components are defined coherently with respect to the self-similar model by the intermediate of extraction filters $w_j(n_1,n_2)$,

$$\tilde{f}^b_{i_1 i_2}(x;y) = \sum_{j\geq J} \sum_{n_1,n_2} d_j^{f,b}(n_1 - 2^j i_1, n_2 - 2^j i_2)$$
$$.w_j(n_1,n_2)\, \Psi^b_{j,n_1,n_2}(x;y) \qquad (4)$$

A piecewise self-similar model can then be identified for each localized function $\tilde{f}^b_{i_1 i_2}(x;y)$. The main change brought to the global model lies in the use of a widest definition of self-similarity which allows to use a different localized function as source information. This means that any local function $\tilde{f}^b_{i_1 i_2}(x;y)$ can be written as

$$\tilde{f}^b_{i_1 i_2}(x;y) = \sum_{l_1,l_2} \alpha^b_{i_1 i_2}(l_1,l_2)\, \tilde{f}^b_{k_1 k_2}(2x - l_1; 2y - l_2)$$
$$+ \tilde{r}^b_{i_1 i_2}(x;y) \qquad (5)$$

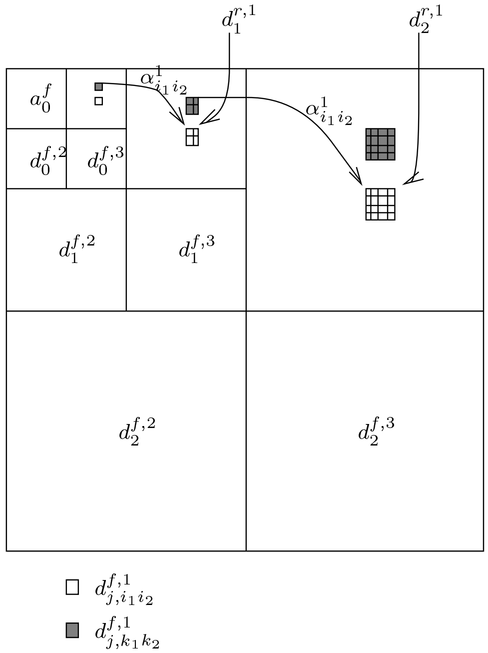

Figure 1: Illustration of the interpolation process

The local function $\tilde{f}^b_{k_1 k_2}(x;y)$ is called the source function of $\tilde{f}^b_{i_1 i_2}(x;y)$. In the latter case, it can be shown that the detail coefficients of $\tilde{f}^b_{i_1 i_2}(x;y)$ and its source function are related to each other through the following relationship

$$d^{f,b}_{j+1,i_1 i_2}(n_1,n_2) = d^{r,b}_{j+1,i_1 i_2}(n_1,n_2)$$

$$+\frac{1}{2}\sum_{l_1,l_2} \alpha^b_{i_1 i_2}(l_1,l_2)\, d^{f,b}_{j,k_1 k_2}(n_1 - 2^j l_1, n_2 - 2^j l_2) \qquad (6)$$

It is worth noting that the coefficients linking the different resolution levels of the two functions do not change from resolution to resolution and correspond exactly to the parameters of the spatial self-similarity definition. In addition to that, the relationship is valid whatever the wavelet basis is, the equation is independent from the wavelet multiresolution analysis and only depends on the self-similar model defined earlier.

3.3. Improvement of the model

The piecewise self-similar model that has been defined can be improved by the introduction of a more complex transformation of the source function. For instance, simple geometrical transformations as rotations or symmetries of the source function can be introduced and their effects can be translated onto the wavelet coefficients. If the target local component $f^b_{i_1 i_2}(x;y)$ has $\Gamma_{i_1 i_2}\left(f^{b'}_{k_1 k_2}(x;y)\right)$ as source function, where $\Gamma_{i_1 i_2}(\cdot)$ denotes a geometrical transformation. The resulting relationship between source and target wavelet coefficients then becomes

$$d^{f,b}_{j+1,i_1 i_2}(n_1,n_2) = d^{r,b}_{j+1,i_1 i_2}(n_1,n_2)$$

$$+\frac{1}{2}\sum_{l_1,l_2} \alpha^b_{i_1 i_2}(l_1,l_2) \check{d}^{b,j}_{k_1 k_2}(n_1 - 2^j l_1, n_2 - 2^j l_2). \qquad (7)$$

In the last equation, $\check{d}^{f,b}_{j,k_1 k_2}(n_1,n_2)$ represents the wavelet coefficients of $\Gamma_{i_1 i_2}\left(f^{b'}_{k_1 k_2}(x;y)\right)$. When the function $\Phi(x;y)$ is symmetrical, these coefficients can easily be computed from the wavelet coefficients of $f^{b'}_{k_1 k_2}(x;y)$ for simple geometrical transformations. A 90° rotation of $f^{b'}_{k_1 k_2}(x;y)$ corresponds to the same rotation of its wavelet coefficients and a change of direction and symmetries correspond to the same symmetries applied to the wavelet coefficients.

3.4. Parameters estimation

According to the fractal image representation theory, the parameters $\alpha^b_{i_1 i_2}(l_1,l_2)$ of the self-similar model can be estimated by minimizing the collage error defined as the distance between the function to be coded and the transformation of it under the assumption that the transformation is contractive on a particular metric space. It can be shown that the collage error minimization corresponds, for orthogonal multiresolution analysis, to the minimization of the interpolation error in the frequency domain given by

$$e^b_{i_1 i_2} = \sum_{j \geq J} \sum_{n_1,n_2} (d^{f,b}_{j,1 i_2}(n_1,n_2)$$

$$-\frac{1}{2}\sum_{l_1,l_2} \alpha^b_{i_1 i_2}(l_1,l_2)\, \check{d}^{f,b}_{j,i_1 i_2}(n_1 - 2^j l_1, n_2 - 2^j l_2))^2 \qquad (8)$$

One can observe that the collage error corresponds to the energy of the residues given by

$$\sum_{j \geq J} \sum_{n_1,n_2} \left(d^{r,b}_{j,i_1 i_2}(n_1,n_2)\right)^2$$

The parameters $\alpha^b_{i_1 i_2}(l_1,l_2)$ are then obtained, by the annulment of the partial derivatives $\frac{\partial e^b_{i_1 i_2}}{\partial \alpha^b_{i_1 i_2}(c_1,c_2)}$, as solution of the following system

$$\underline{\underline{r}} \cdot \underline{a} = \underline{s} \qquad (9)$$

Where the matrix $\underline{\underline{r}}$ is defined by

$$r_{pq} = \frac{1}{4}\sum_{b}\sum_{j \geq J}\sum_{n_1,n_2} \check{d}^{f,b}_{j,k_1 k_2}(n_1 - 2^j l_1, n_2 - 2^j l_2)$$

$$.\check{d}^{f,b}_{j,k_1 k_2}(n_1 - 2^j c_1, n_2 - 2^j c_2) \qquad (10)$$

and the vector $\underline{s}$ by

$$s_p = \frac{1}{2}\sum_{b}\sum_{j \geq J}\sum_{n1,n2} d^{f,b}_{j+1,i_1 i_2}(n_1,n_2)$$

$$.\check{d}^{f,b}_{j,k_1 k_2}(n_1 - 2^j c_1, n_2 - 2^j c_2) \qquad (11)$$

In the previous expressions, $p = c_1 L + c_2$ and $q = l_1 L + l_2$. The vector $\underline{a}$ contains the parameters $\alpha^b_{i_1 i_2}(l_1, l_2)$ according to the following correspondence

$$a_{l_1 L + l_2} = \alpha^b_{i_1 i_2}(l_1, l_2) \tag{12}$$

Moreover, it can be shown that the collage error can be expressed in terms of the vectors $\underline{a}$ and $\underline{s}$ and the detail coefficients of $\tilde{f}^b_{i_1 i_2}(x; y)$ via the following expression

$$e^b_{i_1 i_2} = \sum_{j \geq 1} \sum_{n_1, n_2} \left(d^{f,b}_{j, i_1 i_2}(n_1, n_2) \right)^2 - \underline{a}^T \underline{s} \tag{13}$$

Once the self-similarity parameters are known, the residues can be computed according to

$$d^{r,b}_{j+1, i_1 i_2}(n_1, n_2) = d^{f,b}_{j+1, i_1 i_2}(n_1, n_2)$$
$$-\frac{1}{2} \sum_{l_1, l_2} \alpha^b_{i_1 i_2}(l_1, l_2) \, \check{d}^{f,b}_{j, k_1 k_2}(n_1 - 2^j l_1, n_2 - 2^j l_2). \tag{14}$$

4. SYNTHESIS OF THE RESULTS

The image model we have developed allows to combine in a single representation two types of signal modelization techniques. On one hand, self-similar information is taken into account via the parameters $\alpha^b_{i_1 i_2}(l_1, l_2)$ of the model. On the other hand, the low frequency information represented by the approximation and detail coefficients $a^f_J(n_1, n_2)$ and $d^{f,b}_J(n_1, n_2)$ and the residues given by $d^{r,b}_j(n_1, n_2)$ bring the complementary information to the self-similar one in order to represent the signal without loss of information. The way to estimate the self-similarity parameters ensures to maximize the amount of information contained in the fractal part of the model.

5. TARGETED APPLICATIONS

The image representation technique that has been presented is intended to be used in several applications in the image processing field.

At first, the model coefficients can be quantized and entropy coded in order to lead to an image compression method taking advantages of both frequency and fractal transformations in a locally adaptive way. The performances of this technique should overcome the performances of multiresolution decomposition methods because, in the areas where the self-similarity is low, all the energy can be transfered into the residues and the model then reduces to a multiresolution decomposition while in the highly self-similar zones, nearly all the information is taken into account by the self-similar part of the model.

The fractal-multiresolution representation can also be used to create images with super-resolution. The parameters can be estimated and the inter-scale interpolation process can be conducted beyond the original resolution of the image to create non-existant detail coefficients.

It is also intended to be used in texture modelization and analysis.

6. CONCLUSIONS

We have presented in this paper a new representation technique for images combining fractal and frequency decompositions. The fractal transformation is characterized by an overlap between the self-similar components of the signal and its effects can be translated into the frequency domain thanks to the wavelet multiresolution representation of the signal. The theoretical basis of the modelization technique have been exposed for bi-dimensional signals and potential applications have been envisaged.

7. REFERENCES

[1] Michael F. Barnsley. *Fractals Everywhere.* Academic Press, San Diego, 1988.

[2] Kai Uwe Barthel, Jörg Schüttemeyer, Thomas Voyé, and Peter Noll. A new image coding technique unifying fractal and transform coding. In *IEEE International Conference on Image Processing (ICIP '94)*, pages 112–116, Austin, Texas, November 1994.

[3] Yuval Fisher, editor. *Fractal Image Compression: Theory and Application.* Springer-Verlag, New York, 1995.

[4] Arnaud E. Jacquin. Image coding based on a fractal theory of iterated contractive image transformations. *IEEE Transactions on Image Processing*, 1(1):18–30, January 1992.

[5] Stéphane Mallat. A theory for multiresolution signal decomposition: The wavelet representation. *IEEE Transaction on Pattern Analysis and Machine Intelligence*, 11(7):674–693, July 1989.

[6] B. Simon. Explicit link between local fractal transform and multiresolution transform. In *Proceedings of Second IEEE International Conference on Image Processing*, pages 278–281, October 1995.

[7] B. Simon. Fractal multiresolution transformations and the wavelet representation. *Special Issue of Fractals*, to appear.