EXPLICIT LINK BETWEEN LOCAL FRACTAL TRANSFORM AND MULTIRESOLUTION TRANSFORM

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ABSTRACT

Much researchs are currently carried out in the field of fractal coding of images. Fractal models try to identify and represent self-similarity relationships within images through the use of spatial transformations of the signal associating shrinking, translation, isometries and scaling. These transformations can be interpreted frequentialy and are particularly suited to an interpretation in the multiresolution decomposition space of functions. In this paper, we study the expression of a particular fractal transform within the orthogonal multiresolution context. Conclusions drawn from the results show that the fractal transformation can be decoupled in two terms, one serves to interpolate high frequency coefficients from one resolution layer to the next while the other is a constant term containing low pass information. A fractal transformation taking these elements into account is defined and it is shown that parameters can be estimated only on the basis of multiresolution decomposition coefficients. The way to apply these results to images is then envisaged.

1. INTRODUCTION

In the last few years, fractal coding of still images has become more and more popular among the image coding community. The initial works of Barnsley [1] and Jacquin [3] were based on self-similarity within images. The image to be coded is partitioned into non overlapping blocks (called range blocks), each of them being then described as a contractive transformation of another image block (called domain block). The contractive transformation is made of a set of isometries and a contraction of the luminance profile to which a constant is added.

Concurrently to fractal coding, subband and multiresolution codings [4] continue to evolve towards several directions such as spatial variations of the filters [2], signal adapted filters, etc .

Although it is commonly admitted that fractal coding creates relationships between subband-multiresolution signals [7], no explicit link between these two areas of still image coding has been explicitly demonstrated.

This paper presents an explicit link between a particular type of fractal transform and multiresolution decomposition of the signal. The fractal transform can therefore be interpreted in the multiresolution formalism, and consequently be adapted in order to merge the two methods in a single one exploiting the advantages of both.

2. FRACTAL TRANSFORM DEFINITION

In order to develop the link between fractal coding and multiresolution decomposition, a local fractal transform is defined. A particular case of this transform is a classical fractal transform for which each range block is contained in the corresponding domain block. Mathematical expressions are here developed for unidimensional signals, however the conclusions can be easily extended to bidimensional functions and to range blocks not contained in the corresponding domain block.

Let $f(x)$ be a function of $L^2(R)$, the set of square integrable functions on the 1-D real space. This function is said to be self-similar if it can be described by the following relationships [1]:

$$f(x) = \sum_{c=0}^{L-1} \alpha_c f(2x-c) + \sum_{c=0}^{L-1} \beta_c \Phi(2x-c) \quad (1)$$

$$f(x) = F(f(2x)) + T(2x) \quad (2)$$

[1]when these expressions are used in modelization processes, equalities are replaced by approximation symbols

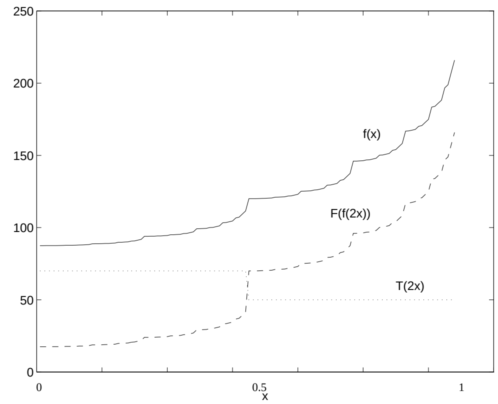

Figure 1: Example of self-similar function

The function $\Phi(x)$ is the scaling function corresponding to a particular orthogonal multiresolution transform and L is the length of the low pass filter corresponding to the given transform. A sufficient condition for that fractal transform to be contractive is given by

$$\sum_c \alpha_c^2 \leq 2. \tag{3}$$

If the transformation is contractive, the function $f(x)$ is obtained as its fixed point and the parameters α_c and β_c can be estimated by using the collage theorem [1]. An example of this kind of transformation is illustrated in figure 1 in the case of Haar multiresolution transform. In this particular case, the function $\Phi(x)$ is the indicatrice function on $[0;1]$ and $L = 2$ leading to a local block decomposition already described in the literature [5] [6]. In all other cases, there will be an overlap between the compressed copies of the signal.

3. LINK BETWEEN THE FRACTAL TRANSFORM AND MULTIRESOLUTION DECOMPOSITION OF THE SIGNAL

The function to be described can be decomposed in a multiresolution basis of $L^2(R)$ given by :

$$\{\Phi(x-n)\}_{n\in Z} \text{ and } \{\sqrt{2^m}\Psi(2^m x - n)\}_{m\in N, n\in Z}. \tag{4}$$

where $\Phi(x)$ is the scaling function and $\Psi(x)$ the corresponding orthogonal wavelet. The decomposition of $f(x)$ in that basis is given by

$$\begin{aligned} f(x) &= \sum_n a_{0,n}\Phi(x-n) \\ &\quad + \sum_{m\geq 0}\sum_n d_{m,n}\sqrt{2^m}\Psi(2^m x - n) \end{aligned} \tag{5}$$

Coefficients $a_{0,n}$ are the low pass approximation of the signal and coefficients $d_{m,n}$ represent the detail information corresponding to higher resolution information of the signal. Functions $\Phi(x)$ and $\Psi(x)$ are defined by

$$\Phi(x) = \sqrt{2}\sum_n h_n \, \Phi(2x-n) \tag{6}$$

$$\Psi(x) = \sqrt{2}\sum_n g_n \, \Phi(2x-n) \tag{7}$$

The filters h_n and g_n are respectively the low pass and high pass filters associated to the multiresolution transform. The decomposition of $f(x)$ in the multiresolution basis (5) can be introduced in (1) in order to express the links joining the coefficients of the multiresolution decomposition in the case of a self-similar function. Before presenting the results, we need to state some definitions.

The filters p_n and q_n are defined by the following expressions:

$$p_n = \frac{1}{\sqrt{2}}\sum_l \alpha_l \, h_{l-n} \tag{8}$$

$$q_n = \frac{1}{\sqrt{2}}\sum_l \alpha_l \, g_{l-n} \tag{9}$$

and filters u_n and v_n are related to the coefficients β_c of the fractal transform in the following way:

$$u_n = \sqrt{2}\sum_l \beta_l \, h_{n-2l} \tag{10}$$

$$v_n = \sqrt{2}\sum_l \beta_l \, g_{n-2l} \tag{11}$$

Theorem 1 *A function $f(x)$ of $L^2(R)$ presenting self-similarity in the sense of (1) fulfills the following properties between its multiresolution decomposition coefficients:*

$$a_{0,n} = \sum_l a_{0,l} \, p_{2n-l} + u_n \tag{12}$$

$$d_{0,n} = \sum_l d_{0,l} \, q_{2n-l} + v_n \tag{13}$$

$$\vdots$$

$$d_{i+1,n} = \frac{1}{\sqrt{2}}\sum_l d_{i,n-2^i l} \, \alpha_l \tag{14}$$

$$\vdots$$

The results obtained in this section present several interesting properties. The influence of the constant term of the fractal transform (i.e. $T(2x)$) is restricted to the low pass approximation ($a_{0,n}$) and to the first detail coefficients ($d_{0,n}$) of the multiresolution decomposition of the signal. The detail coefficients for a scale higher than 0 ($d_{i+1,n}$) are interpolated from the nearest lower resolution ($d_{i,n}$) by the use of a filter. This filter is constituted by α_c interpolated by a factor 2^i for the detail coefficient at resolution $i+1$.

The two terms of the fractal transform play distinct roles on the multiresolution decomposition of the signal. The constant term ($T(2x)$) is there to represent the low resolution information of the signal while the contractive term ($F(2x)$) creates the inter-resolution relationships through the use of its coefficients α_c. The initiator of the inter-resolution interpolation is the first layer detail coefficients ($d_{0,n}$) and is also represented by the constant term $T(2x)$ of the fractal transformation.

These two remarks can therefore lead us to define a new fractal transform within the framework of multiresolution decomposition of images characterized by a perfect transmission of the low pass and the first detail coefficient, while the other detail coefficients are interpolated from one resolution to the other by the use of a filter estimated by minimizing a collage error.

4. COMBINATION OF MULTIRESOLUTION DECOMPOSITION AND FRACTAL TRANSFORM

The goal of this section is to define a transformation combining multiresolution decomposition of a signal and a fractal transform defined on this signal. The previous section conclusions suggest to compute the iterations of the fractal transform directly in the transform domain. Moreover it is shown that the fractal coefficients (α_c, β_c) can be estimated only by using the coefficients of the multiresolution decomposition of the signal. The fractal transform will also act locally due to the splitting of the function $f(x)$ into a sum of more localized functions $f_i(x)$.

4.1. Definition

It can be shown that any signal $f(x)$ can be decomposed as a sum of localized translated blocks

$$f(x) = \sum_i f_i(x-i) \quad (15)$$

where the functions $f_i(x)$ are given by

$$\begin{aligned} f_i(x) &= a_{0,i}\Phi(x) + d_{0,i}\Psi(x) \\ &\quad + \sum_{m>0}\sum_n \bar{d}^i_{m,n}\sqrt{2^m}\Psi(2^m x - n) \quad (16) \\ f_i(x) &= a_{0,i}\Phi(x) + \bar{f}_i(x) \quad (17) \end{aligned}$$

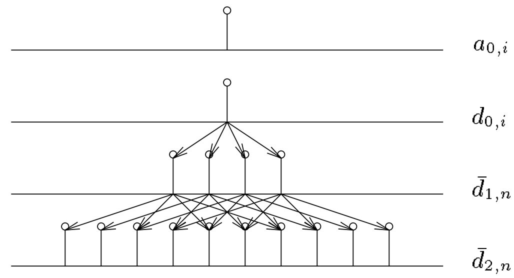

Figure 2: Illustation of inter-resolution interpolation process

Expression (16) is derived from (5) by keeping one approximation and one detail coefficient, higher detail layer coefficients ($\bar{d}^i_{m,n}$) are deduced from detail coefficients $d_{k,l}$ of the function $f(x)$ through a windowing by a filter depending on the underlying multiresolution analysis. In following expressions, indices appearing in summations on $\bar{d}^i_{m,n}$ are assumed to be limited to that window. A fractal transform $FR(\cdot)$ is defined on $\bar{f}_i(x)$ by :

$$FR(\bar{f}_i(x)) = \sum_{c=0}^{L-1} \alpha^i_c \bar{f}_i(2x-c) + d_{0,i}\Psi(x) \quad (18)$$

This last expression is to be compared with (1), as the function $\bar{f}_i(x)$ contains only high frequency information, the constant term of the fractal transform is reduced to its first detail coefficient. This allows to decouple the effects of terms $F(f(2x))$ and $T(2x)$ of (2) in the fractal transform. The contractivity condition is the same as (3). The computation of the successive iterations of the fractal transform can be done in the multiresolution transform domain thanks to the following relationship:

$$\bar{d}^i_{j,n} = \frac{1}{\sqrt{2}}\sum_l \bar{d}^i_{j-1,n-2^{j-1}l}\alpha^i_l. \quad (19)$$

The inter-resolution interpolation process is illustrated in figure 2 for a multiresolution analysis having filters of length $L=4$.

4.2. Parameters estimation

The parameters α^i_c are estimated by minimizing the collage error e_i

$$e_i = \int \left(\bar{f}_i(x) - \sum_{c=0}^{L-1}\alpha^i_c \bar{f}_i(2x-c) - d_{0,i}\Psi(x)\right)^2 dx \quad (20)$$

Annulment of derivatives $\frac{\partial e_i}{\partial \alpha^i_c}$ leads to the following matricial equation

$$R^i \cdot \alpha^i = S^i \quad (21)$$

where matrices R^i and S^i are defined by

$$r^i_{np} = \frac{1}{2} \sum_{m \geq 0} \sum_{l} \bar{d}^i_{m,l-2^m n} \; \bar{d}^i_{m,l-2^m p} \qquad (22)$$

$$s^i_n = \frac{1}{\sqrt{2}} \sum_{m>0} \sum_{l} \bar{d}^i_{m,l} \; \bar{d}^i_{m-1,l-2^m n} \qquad (23)$$

Matricial equation (21) can be seen as the dual operation of expression (14) linking the detail coefficients through α_c parameters, in (21) parameters α_c are obtained as a combination of these detail coefficients.

The computation of the fractal transform coefficients is consequently done only by using the detail coefficients of the signal multiresolution decomposition.

5. APPLICATION TO IMAGES

The samples of a discretized signal are directly related to the multiresolution transform of the signal via a subband decomposition. The difference in this case is due to the fact that detail coefficients are only known up the initial resolution of the image. Consequently, coefficients α^i_c will be estimated on the base of this available information.

In the case of images, the multiresolution transform is computed in a separable way and the developments presented in the paper remain valid. The coded information is composed of the low pass approximation and the three high pass subbands of the lower resolution. The higher frequency subbands are directely interpolated via the fractal transformation in the multiresolution decomposition domain. In this case, filters are bidimensional. In a practical way, the shortest filters are Haar filters ($L = 2$) and will correspond to work with block self-similarity. The use of longer filters will decrease blocking artifacts but at a higher computational cost.

Applications to be envisaged with such fractal transforms in the multiresolution domain are the coding of images with a high compression ratio by taking into account the human visual system capacities. This can be achieved by modifying lightly the fractal transformation previously defined and by introducing a frequential weighting factor in the expression of the collage error. Another important property of the presented scheme is the possibility to increase the image resolution artificially by interpolation of non-existing subbands through the use of filters estimated in minimizing the collage error.

6. CONCLUSIONS

We have presented in this paper a explicit link between a particular class of fractal transform and a multiresolution decomposition of signals. The interpretation of the results points out that the low pass components of the signal are approximated by the constant term of the fractal transform ($T(2x)$) and the contractive term ($F(f(2x))$ is responsible for the successive higher details interpolation. Based on these observations, a new transfom combining multiresolution decomposition and fractal transform has been introduced. In this transform, all computations can be achieved exclusively by using the multiresolution decomposition coefficients.

The next issue will be to compare this technique with both purely fractal methods and multiresolution decomposition of images. The possibility of zooming the signal will also be envisaged by interpolating non-existing high resolution subbands through the use of filters estimated in the analysis part.

7. REFERENCES

[1] M. F. Barnsley. *Fractals Everywhere.* Academic Press Inc., 1988.

[2] C. Herley and M. Vetterli. Orthogonal time-varying filter banks and wavelet packets. *IEEE Transactions on Signal Processing*, 42(10), October 1994.

[3] A. E. Jacquin. Image coding based on a fractal theory of iterated contractive image transformations. *IEEE Transactions on Image Processing*, 1(1):18–30, 1992.

[4] Stéphane Mallat. A theory for multiresolution signal decomposition: The wavelet representation. *IEEE Transaction on Pattern Analysis and Machine Intelligence*, 11(7):674–693, July 1989.

[5] D. M. Monro. A hybrid fractal transform. In *Proceedings of ICASSP'93*, pages V: 169–172, 1993.

[6] D. M. Monro and S. J. Woolsey. Fractal image compression without searching. In *Proceedings of ICASSP'94*, pages V: 557–560, 1994.

[7] R. Rinaldo and G. Calvagno. Image coding by block prediction of multiresolution subimages. *IEEE Transactions on Image Processing*, to appear, July 1995.