A Jointly Optimal Fractal/DCT Compression Scheme

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***Abstract*—In this paper a hybrid fractal and Discrete Cosine transform (DCT) coder is developed. Drawing on the ability of DCT to remove inter-pixel redundancies and on the ability of fractal transforms to capitalize on long-range correlations within the image, the hybrid coder performs an operationally optimal, in the rate-distortion sense, bit allocation among coding parameters. An orthogonal basis framework is used within which an image segmentation and a hybrid block-based transform are selected jointly. The selection of coefficients in the DCT component of the overall block transform is made a part of the optimization procedure. A Lagrangian multiplier approach is used to optimize the hybrid transform parameters together with the segmentation. Differential encoding of the DC coefficient is employed, with the scanning path based on a 3rd-order Hilbert curve. Simulation results show a significant improvement in quality with respect to the JPEG standard, an approach based on optimization of DCT basis vectors, as well as, the purely fractal techniques.**

I. Introduction

FRACTAL image coding takes advantage of image self similarities on different scales. The fractal method of compression is based on the observation that there exists a class of artificial images, such as Mandelbrot and Julia sets [1], which are rich in detail, but contain little information in the sense that they can be generated by the recursive application of scale-varying transformations to some simple initial images. The idea of using image self-similarities to achieve compression is first attributable to Barnsley [1].

Mathematically, if x is the image to be encoded, a transformation T is sought such that $x = x_T$, where

$$x_T = \lim_{k\to\infty} T^k x_0 \tag{1}$$

for any arbitrary initial image x_0. For a certain class of transformations T, this limit exists, in which case x_T is called the attractor of T. It is also called the fixed point of T because x_T is invariant under T, i.e.,

$$x_T = Tx_T. \tag{2}$$

Compression is achieved due to the fact that usually an initial image x_0 together with a transformation T require fewer bits to be described than the grey-level image directly. Application of fractal methods to natural images, however, is not straight-forward, since the self-similarity assumption is not always justified, at least not under simple affine transformations. An arbitrary image is not guaranteed to possess such a constructive transformation at all and, if it does, the transformation may require more bits to be described than the original image. Fortunately, in many practical applications, some reconstruction error is tolerated. With e the error between the original and reconstructed images, a lossy version of the encoding is

$$x = \lim_{k\to\infty} T^k x_0 + e \tag{3}$$

for any arbitrary initial image x_0. When the transformation is applied to the original image x, the output, Tx, is called the collage of x under T. A metric d, usually the square norm, is chosen to quantify fidelity of the reconstructed image to the original. As shown in [6], the collage error, defined as $d(x, Tx)$, can be used to estimate the reconstruction error, which is used in the encoding process as the objective function to be minimized. Optimality of fractal transform coefficients, using the collage theorem, was investigated in [9].

Most fractal algorithms, beginning with Jacquin's implementation [10], operate on a segmented image consisting of N nonoverlapping square regions r_i, called ranges, with $x = \sum_{i=1}^{N} r_i$. In order to avoid certain artifacts associated with square segmentation, alternative (triangular, hexagonal) segmentation schemes have been investigated [4], [7], [20]. The block-based segmentation, nevertheless, remains the most popular approach. In a fractal coder, each square range block r_i is encoded by a nonexpansive transformation $T_i (T = \sum_{i=1}^{N} T_i)$ operating on the whole image and mapping a domain block, d_i, twice the size of the range block, onto r_i. For each range block, the encoder seeks the transformation T_i operating on the whole image x, minimizing the collage error $\|r_i - T_i x\|$, which is much simpler to do for one range region at a time than for the entire x at once. The collage theorem guarantees an upper bound on the reconstruction error as a function of the collage error and the contractivity of T_i [6].

Unlike transform coding methods, where the goal is to decorrelate pixels in an image region by taking advantage of intra-region redundancies, the fractal method addresses redundancy on the region-to-region basis. In other words, the premise is that for every range region, we can find a contractive transformation operating on a different part of the image, which results in a close approximation of the range under consideration. This also means that the amount of distortion in the decoded image is dependent on how much self-similarity there is present in the given image. Description of the contractive transformation T, therefore, constitutes a lossy code for image x.

The structure of each transformation is fixed and consists of a decimation operator, an isometry, a multiplication by a scalar, and addition of an intensity translation block. Alternatively, some range blocks can be encoded by a translation block only. The function of the encoder is to make an optimal selection among all possible combinations of these parameters, henceforth referred to as quantizers. Cardinality of the set of quan-

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tizers determines how computationally expensive a given fractal algorithm is. In the proposed approach, this cardinality is limited through the use of coarse quantizers for domain-range displacement, the scaling parameter, and by restricting the number of isometries.

Compression achieved by a fractal code is directly related to the number of transforms. The fewer regions the image is partitioned into, and the shorter the description of each transformation the higher, the compression ratio. On the other hand, large range blocks are less likely to be found elsewhere in the image on a different scale, and the coarse quantization of the transform parameters leads to larger collage distortion and, ultimately, larger error energy at the decoder. With the proposed method, we let the encoder select the most efficient image partition, based on the quad-tree (QT) structure, and the most efficient quantizer for each block, based on the target rate-distortion tradeoff for the entire image.

Blocks for which a good approximation, under a contractive transformation, can be found elsewhere in the image, can be efficiently coded using a fractal transform, which is designed to take advantage of long-range correlations within an image. The self-similarity assumption which is central to fractals, however, may not be justified for all blocks. In these cases, spending more bits on the fractal transform, by employing more isometries or finer quantizers is not very efficient [7], [19], [14].

The Discrete Cosine transform (DCT) has been the transform of choice for most codecs due to its decorrelation and energy compaction properties. Operating on a single block, it can be thought of as capitalizing on short-range correlations within an image. Complicated image features, however, require a significant number of DCT coefficients for good fidelity of representation. The coarse quantization of the DCT coefficients in this case results in blocking artifacts and unsharp edges.

The complementing nature of the fractal and the discrete cosine transforms suggests their joint use when the task at hand is the maximal removal of redundancies in an image. One way to integrate the two within the same codec is by assigning partitioned blocks to either of the two transforms. The proposed method goes much further by allowing both transforms to operate on the same block in an orthogonal setup. It combines fractal based coding with DCT coding and within the chosen algorithmic structure, arrives at an optimal code.

In the proposed approach all coding takes place in the frequency (DCT) domain. It is the function of the encoder, for each range block, to determine the position of the DCT coefficients treated as nonzero, i.e., coded using the (*zerorun*, *magnitude*) mechanism, employed by JPEG. Alternatively, we also investigate the effectiveness of using a fixed number of predetermined nonzero DCT coefficient sets, which can be described by indexing. In both cases, the remaining range DCT coefficients are then approximated by their DCT-transformed domain block counterparts or by zeros, depending on the Lagrangian cost.

The Lagrangian multiplier method is used in Section II to convert the constrained problem of finding the hybrid code and segmentation which minimize the distortion measure for a given bit budget, into an unconstrained minimization problem. Section III describes how fractal and DCT transforms can be combined in the encoder, based on either dynamic or fixed DCT coefficient sets. The use of inter-block predictive coding in the algorithm couples code selection and segmentation decisions among blocks. A dynamic programming first-order backward dependency scheme, described in Section IV, is used to arrive at the optimal sequence of quantizers in a quad-tree (QT) decomposition. Finally, experimental results and conclusions are presented in Sections V and VI, respectively.

II. Problem Formulation

The problem at hand is to simultaneously segment an image into blocks of variable sizes and, for each, to find a hybrid code such that any other choice of segmentation and coding parameters would result in a greater distortion for the same rate, or, alternatively, higher rate for the same distortion. Mathematically, for a given image x, we want to solve the following optimization problem:

$$\min_{s\in S, c\in C_s} D(x_{s,c}, x) \quad \text{subject to} \quad R(x_{s,c}) \leq R^* \tag{4}$$

where $x_{s,c}$ is the encoded image, D the distortion metric, s a member of the set of all possible image segmentations S, c a member of C_s, the set of all possible codes given segmentation s, $R(x_{s,c})$ the bit rate associated with segmentation s and code c, and R^* the target bit budget. The distortion metric chosen here is the l_2-norm. The constrained optimization problem stated in (4) is converted into an unconstrained problem using the Lagrangian multiplier method. That is the following problem is solved

$$\min_{s\in S,\, c\in C_s} G(x_{s,c}, \lambda) = \{D(x_{s,c}, x) + \lambda \cdot R(x_{s,c})\}. \tag{5}$$

The multiplier $\lambda = \lambda^*$, with corresponding optimal segmentation s^* and code c^*, for which $R(x_{s^*,c^*}) \approx R^*$, can be efficiently found using the monotonicity of the operational rate-distortion (ORD) curve, as in [21]. The same framework is also suitable for solving the dual problem when the average image distortion is constrained and the rate is to be minimized, in which case R and D in (5) must be interchanged. Using this formulation, we overcome a disadvantage of conventional fractal coders which is their inability to provide good rate control when high fidelity is required. It stems from the fact that allocating more bits per transformation, or in other words, by allowing a greater domain pool or more isometries, after a certain point, does not lead to increased reconstruction quality. With the above formulation, on the other hand, the rate is inversely related to λ, and thus can be controlled.

III. The Hybrid Transformation

The hybrid transformation, proposed in this paper, aims to exploit both short-term and long-term correlations present in a typical image. The stated objective is achieved by making the two components (fractal and DCT) orthogonal to each other and by adapting the partition to the image. Among all possible partitions and block quantizers the encoder selects the image partition and the appropriate balance of DCT and fractal transform per block, such that the overall problem, expressed in (4), is solved.

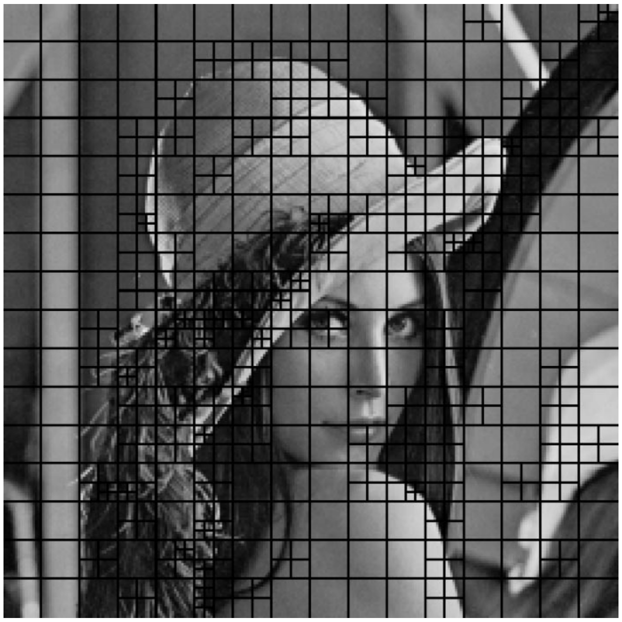

Fig. 1. Optimal segmentation ($R = 0.45$ bpp, $PSNR = 30.33$ dB).

A. Segmentation

The set of all possible segmentations S is restricted to be on the quad-tree lattice as a compromise between local adaptivity and simplicity of description. It ensures that the segmentation overhead is low, since only 1 bit is required to indicate whether a parent square block is split into four subblocks, with no such bit required for blocks of the smallest predetermined size. Keeping all ranges square makes the application of fractal and frequency domain methods straightforward. Without lack of generality, for a 256 $\times$ 256 image we use a 3-level quad-tree with the maximum and minimum block sizes of 16 $\times$ 16 pixels and of 4 $\times$ 4 pixels, respectively. Fig. 1 demonstrates a QT partition of the *Lena* image that resulted from the application of the proposed algorithm with target rate of 0.45 bpp.

B. Code Structure

Making components of the fractal transform orthogonal to each other carries many benefits [15]. Among these are fast convergence at the decoder, noniterative determination of scaling and intensity translation parameters, and removal of the magnitude restriction on the scaling coefficient for convergence. Here we explore another such benefit, the local continuity of the DC value of the intensity translation term. Frequency domain interpretation lends itself naturally to the concepts of orthogonality and energy compactness, which is why, as in [26], we perform the collage error minimization in the DCT domain. In agreement with the terminology in [17] and [16], we use the vector space representation of domains, ranges and translation terms.

Let us first denote by x the original image. The $M^2 \times 1$ vector d_i, henceforth referred to as domain vector, is obtained by the fractal transformation of the domain block through the application of the following sequence of operators:

$$d_i = P_i \; Z_i \; DCT_i \; D_i \; I_i \; Fet_i \; x \tag{6}$$

where Fet_i is the "fetch the correct domain block" operator, I_i the isometry operator, D_i the spatial decimation by 2 operator, DCT_i the 2-D DCT transform of size $M \times M$, Z_i the zigzag scan operator transforming a block into a vector, and P_i the "put in the correct location" operator.

Let us also denote by an $M^2 \times 1$ vector r_i the zigzag scanned DCT coefficients c_{ik}, $k = 0, \ldots, M^2 - 1$ of an $M \times M$ range block. The objective now is to provide a lossy approximation of each such range block of the original image given an image partition. We will use the $N - 1$ low frequency coefficients in approximating the block but will replace the remaining coefficients by the corresponding coefficients in d_i in (6). More specifically the vector r_i is approximated by

$$r_i \approx \beta_i \cdot d_i^p + \sum_{k=0}^{N-2} c_{ik} \cdot f_{ik} \tag{7}$$

where d_i^p is the projection of d_i onto the orthogonal complement of the subspace spanned by the mutually orthogonal vectors f_{ik}, for $k = 0, \ldots, N-2$, and β_i are scalars. It should be noted that in general f_{ik} can represent any orthogonal basis, in which case c_{ik} are the projection coefficients in that base.

With the code structure defined by (6) and (7), the task of the encoder is to optimally select, for the range vector r_i, operators Fet_i and I_i, and the subspace (with its dimension $N-1$) defined by vectors f_{ik}. Clearly, it is desirable to keep $N-1$, the number of basis vectors f_{ik}, as small as possible if a high compression ratio is desired. Note that the other operators in (6) are purely deterministic, i.e., they do not involve any decision making by the encoder. Similarly, the scalars c_{ik}, $k = 0, \ldots, N-2$, are found deterministically as lengths of projections of r_i onto the basis vectors f_{ik}.

A noniterative procedure for finding β_i in (7) that yields a least-squares approximation of r_i in an N-dimensional vector space is given in [17]. A similar approach was tried in [8] in which multiple orthogonalized domain vectors were allowed. DCT domain modeling of range vectors was used in [2], but that approach suffered from the lack of orthogonality of d_i to vectors f_{ik}, a prescribed uniform partition, a restriction placed on the scaling coefficient β_i, and an independent encoding of all transform coefficients.

In the following two sections, we describe two alternative methods for defining the DCT component of the overall transform. One is based on the idea of using an indexed bank of fixed subspaces f_{ik}, and the other allows more flexibility at the encoder by employing the JPEG model to specify the retained coefficients (bases), per block.

C. Fixed DCT Subspace

In this section, we describe an approach based on the use of a bank of fixed subspaces [11], [13], generated by DCT coefficients, to model a range vector of a given size. To illustrate how a fixed subspace is formed, let us, for simplicity, consider a 2 $\times$ 2 block of DCT coefficients. Let us also assume that the lower three of these coefficients are used to form a subspace of dimension 3, when the full space is of dimension 4. Each f_{ik} corresponds to one coefficient in the two-dimensional DCT of

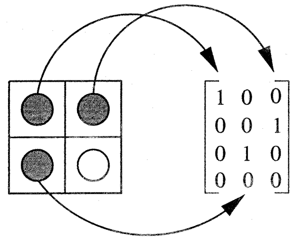

Fig. 2. Mapping of selected DCT coefficients into basis vectors.

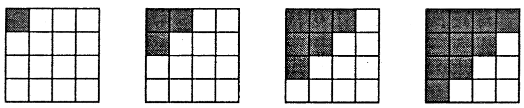

Fig. 3. Four subspaces for block size of 4.

size $M \times M$, as shown in Fig. 2 for $M = 2$. Each vector f_{ik} has zeros in all positions, except the one corresponding to the zigzag ordered location of the DCT coefficient it represents, where it has the value of one. Generally, in larger blocks more DCT coefficients tend to be significant. Hence, the fixed space used for the coding of a 16 $\times$ 16 range block will be allowed to be of a higher dimension than that of a 4 $\times$ 4 block. Limiting the number of available subspaces has the advantage that only its index needs to be sent to the decoder, whereas in [2] individual coefficient positions had to be sent as well. Although in general one could optimally design the fixed subspaces to be used, it is not addressed in this section, because it is done adaptively (i.e., for a nonfixed subspace) in Section III-D. As an example, shown in Fig. 3 are the subspaces used in this work for block sizes 4 $\times$ 4. They are created using respectively the 1, 2, 3, and 4 45° diagonals. The function of the fractal component of the overall transform, expressed through $\beta_i \cdot d_i^p$ in (7), is to approximate all the remaining DCT coefficients. A bank of subspaces with different numbers of retained coefficients is equivalent to an ordered set of quantizers, ranging from the coarsest to the finest. The encoder thus can achieve local adaptivity by selecting, for a given block, the subspace resulting in the desired rate-distortion tradeoff. Therefore, relatively flat regions or regions well approximated through the fractal component will require a subspace of a small dimension.

The subspaces for block sizes 8 $\times$ 8 are generated similarly from the 1, 3, and 4 45° diagonals, and for block sizes 16 $\times$ 16, the maximum allowable size in a partition, from the 1, 4, and 5 diagonals, respectively. Thus we allow energy carrying low-frequency DCT coefficients to provide the subspace vectors f_{ik}. For each domain vector, only the component orthogonal to the subspace spanned by f_{ik} is used to approximate a given range block. Making d_i orthogonal to this subspace is trivial and is accomplished by setting to zero those positions of d_i that correspond to the retained f_{ik}. Clearly, here the subspace dimension uniquely identifies the basis vectors used.

The fact that d_i^p and f_{ik}, for $k = 0, \ldots, N-2$, are mutually orthogonal vectors can be used to compute in general the least squares solution of (7) for the scaling coefficient β_i and weights c_{ik} [17]. In our formulation, the latter are simply the quantized magnitudes of the corresponding range block DCT coefficients. The coefficient β_i is computed by [17]

$$\beta_i = \frac{\langle d_i^p, r_i - F_i c_i \rangle}{\langle d_i, d_{oi} \rangle} \tag{8}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product, d_{oi} is the projection of d_i onto the subspace complement to that spanned by vectors f_{ik}, $c_i = [c_{io}, \ldots, c_{i(N-2)}]^T$ and $F_i = [f_{i0}, f_{i1}, \ldots, f_{i(N-1)}]$. Issues related to the encoding of parameters describing the fractal and DCT components of the overall transform, including β_i and f_{ik}, are discussed in Section III-E.

D. Optimized DCT Subspace

In this section we describe an algorithm for optimally selecting a set of DCT coefficients forming the subspace, defined by f_{ik}, in (7). In the previous section we described a method of selecting the DCT component based on the idea of using a bank of available subspaces. That approach benefits from simplicity of description (a subspace index uniquely identifies the set of retained DCT coefficients), but suffers from its inherent inability to efficiently model image blocks whose significant DCT coefficients do not follow one of the predefined templates.

We generalize the idea of the previous section by allowing the encoder to retain any sequence of DCT coefficients, while still maintaining its orthogonality to the fractal component. For a known fractal domain vector d_i (6) and a fixed encoding scheme, each particular set of block DCT coefficients results in a rate-distortion pair. The ability to optimally solve the coefficient selection problem for a given d_i is a building block of the proposed method and is necessary if the joint partition, fractal and DCT component optimality is desired.

Clearly, operational optimality in selecting the set of DCT coefficients forming the basis f_{ik}, for $k = 0, \ldots, N-2$, requires specification of the coefficient encoding scheme, as well as, the distortion metric used. Section III-D-1 describes the former—a JPEG-like scheme based on zeros and runs. Having quantified the block distortion by the mean squared error (MSE) metric, we then proceed, in Section III-D2, to cast the problem of optimal coefficient selection as that of finding the shortest path in a Directed Acyclic Graph (DAG).

1) The Rate: The appeal of DCT as a transform of choice for many compression applications has to do with its ability to pack most of the information of an image block into a few low-frequency coefficients. Thus, coarsely quantizing or eliminating altogether the remaining high-frequency coefficients (by setting them to 0) is a reasonable strategy to achieve compression with minimal degradation in quality.

We utilize the JPEG model of encoding DCT coefficients [18] both because of its efficiency and because JPEG is used to benchmark rate-distortion performance of the proposed method. Based on the statistical observation that nonzero AC coefficients are sparse, they are encoded by an aggregate symbol consisting of a run of zeros followed by the magnitude category of a given coefficient. Additionally, a number of bits (equal to the magnitude category) need to be appended to the aggregate symbol to identify the coefficient within the category. Thus, for example, if $a_7 = 28$ and $a_5 = 200$ are two consecutive nonzero AC coefficients, the code for a_7 consists of a run of 1 and category 5, which, according to JPEG defaults and this implementation, requires 11 bits for the aggregate symbol plus 5 additional bits. Hence, the rate required to encode coefficient a_7 following coefficient a_5 is denoted by $r(5, 7)$ and is equal to 16 bits.

Note that the encoding step described here operates on DCT coefficients after quantization, which is what makes high compression ratios possible and makes JPEG a lossy scheme. Based on the statistical observation that after quantization higher-index coefficients in DCT are very likely to be zeros, a special end-of-block (EOB) symbol is incorporated into the symbol alphabet.

One crucial difference between standard JPEG and the proposed method of encoding the DCT components lies in the fact that the latter is not restrained to encode each and every nonzero AC coefficient. Rather, that decision is made based on the global (block) rate-distortion considerations. This feature is especially useful when the complementing fractal component of the transform does a good job of approximating many significant (nonzero) AC coefficients.

2) DAG Formulation: Let $n_0, \ldots, n_{M-1}$, $(M \leq N)$ denote the indices of the set of retained significant nonzero coefficients, henceforth referred to simply as *significant coefficients*. Then the problem to be solved in selecting the optimal sequence of these coefficients for the DCT component of the transform, given the fractal component, can be stated as follows:

$$\min_{n_0, \ldots, n_{M-1}, M} D(n_0, \ldots, n_{M-1})$$

subject to:

$$R(n_0, \ldots, n_{M-1}) \leq R_{\max}^{DCT}, \tag{9}$$

where $D(\cdot)$ is the distortion measure, quantifying the error between the approximation and the original block, $R(\cdot)$ is the bit rate required to encode the M significant coefficients, and $R_{\max}^{DCT}$ is the maximum bit rate permitted for the encoding of the block's DCT component. Note that $R_{\max}^{DCT} = R_{\max} - R^{fractal}$ is the DCT component's bit budget *after* encoding the fractal component with $R^{fractal}$ bits. Note that the above minimization is performed over coefficient indices, as well as, their number M.

The problem of (9) can be posed in terms of finding the shortest path in a DAG, for which efficient solutions are known [3], [22]. Let a_i, for $i = 0, \ldots, N-1$, denote the zigzag-ordered set of DCT coefficients of the block under consideration, with a_0—the DC coefficient. Also, let a_N be the end-of-block (EOB) symbol, terminating encoding of a block, as described in Section III-D1. We associate the ordered set $A = a_i, i = 0, \ldots, N$ with the vertices of a DAG, in which nonnegative directed edges exist between vertices of increasing order. That is, there is an edge from a_j to a_k, if and only if, $j < k$. Fig. 4 illustrates a simple example for $N = 5$ (DCT of a 2 × 2 block), with all valid edges shown by arrows. There is a one-to-one correspondence between a path from vertex a_0 to a_N on one hand and a particular code (in terms of *zero runs* and *levels*, as described in Section III-D1) of the block, on the other.

Let $d(i, j)$ be the squared-error distortion attributable to DCT coefficients $(i+1, \ldots, j)$, i.e., both the retained significant coefficients and the complementing coefficients generated by the fractal component. Note that even though all processing takes place in the DCT domain, Parseval's theorem guarantees equivalence of this metric in the spatial domain. In the case $j = N$, the EOB vertex, calculation of distortion terminates with coefficient $N - 1$. An exception to this rule is the transition from vertex $N - 1$ to N, which is associated with zero rate and zero distortion, in compliance with JPEG.

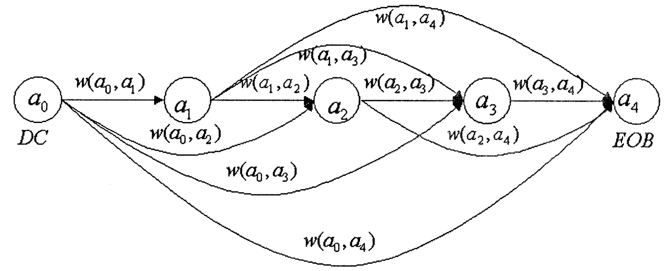

Fig. 4. Simple DAG for selecting the optimal subspace.

The hard to solve constrained minimization problem of (9) is converted into its unconstrained counterpart with the use of the Lagrangian multiplier λ. The unconstrained Lagrangian cost function J_λ^{DCT} can then be expressed as follows:

$$J_\lambda^{DCT}(n_0, \ldots, n_{M-1}) = D(n_0, \ldots, n_{M-1}) + \lambda \cdot R(n_0, \ldots, n_{M-1}). \tag{10}$$

It has been shown in [5], [25] that if there is a λ^* such that,

$$\{n_0^*, \ldots, n_{M-1}^*\} = \arg \min_{n_0, \ldots, n_{M-1}, M} J_{\lambda^*}(n_0, \ldots, n_{M-1}) \tag{11}$$

and which leads to $R(n_0, \ldots, n_{M-1}) = R_{\max}^{DCT}$, then $\{n_0^*, \ldots, n_{M-1}^*\}$ is also an optimal solution to (9).

Besides removing the constraint, the problem formulation of (10) and (11) is more useful because $R_{\max}^{DCT}$ is never given explicitly. On the other hand, it can be shown [12], that the same λ must be used in every subproblem in order to maintain global optimality of the solution. Therefore, the λ in (10) is equal to that of (5).

The (Lagrangian) cost associated with transition from vertex i to vertex j can be expressed as

$$w(i, j) = d(i, j) + \lambda \cdot r(i, j) \tag{12}$$

where $r(i, j)$ is the rate required to encode a_j if the last encoded coefficient was a_i. Then the cost of a path starting with coefficient a_0 and terminating with a_N is equal to

$$\sum_{k=1}^{M} w\left(a_{n_{k-1}}, a_{n_k}\right). \tag{13}$$

Note that the starting and the terminating points are fixed to be the vertex corresponding to the DC coefficient and the EOB symbol, respectively.

Having formulated the optimal significant coefficient selection problem as a shortest path in a DAG, we employ a modified Dijkstra's algorithm [23], of time complexity $\Theta(|V| + |E|)$, to quickly arrive at the solution.

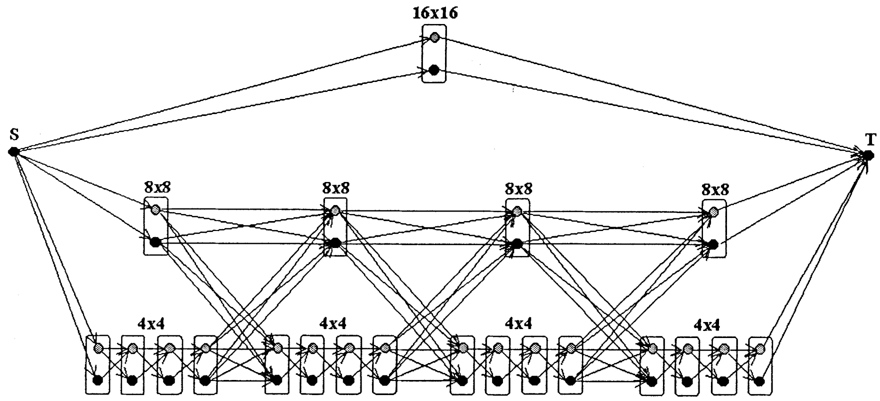

Fig. 5. Optimal code as the shortest path in a trellis.

E. Implementation Issues

In order to employ JPEG's model for encoding DCT coefficients, we used the default 8×8 block quantization table, given in [18]. Quantization tables for 4×4 and 16×16 blocks were obtained from the default one by *ad-hoc* decimation and interpolation. Overall, the obtained rate-distortion curves, shown in Section V, showed very little sensitivity to the quantization tables used.

The relative position of a domain block with respect to the range block was encoded using a variable length code (progressively more bits are used for farther distances). A 32×32 search window centered around the range block, with step size of 2 pixels was used to generate codebooks. Isometries were limited to identity, horizontal and vertical symmetries, and rotation by 180°, requiring 1, 2, 2, and 3 bits, respectively. For the fixed DCT subspace approach the scaling parameter β_i was nonuniformly quantized using a Max–Lloyd quantizer with 5 bits, while with the optimal subspace approach it was kept constant at 0.7, requiring 0 bits. Experiments have shown no significant performance gains from extending the range of allowed β_i, while the complexity was increased. In the fixed DCT subspace approach coefficients c_{ik} [(7)] were encoded using JPEG's variable length codes, except that no zero-runs were used. In that case, the overhead information consisted of 1 or 2 bits to indicate what fixed subspace was used (depending on the block size).

Additionally, a 1 bit flag was included to signal the decoder, in both approaches, whether the fractal component is present, i.e., whether domain vector d_i^p is used as the Nth basis vector in (7). If no fractal component is used, all coefficients complementing the DCT component were set to zero. The decision whether to use the fractal component is included in the overall optimization process.

In order to reduce computational complexity the number of candidate domains for joint fractal/DCT optimization is limited by a preprocessing step. In it, the best L domains are retained per block, based on collage error minimization without the complementing DCT component, i.e., using the standard fractal distortion metric. While, in principle, this preprocessing step introduces suboptimality, experiments have shown that there is very little performance gain if L is increased beyond 100.

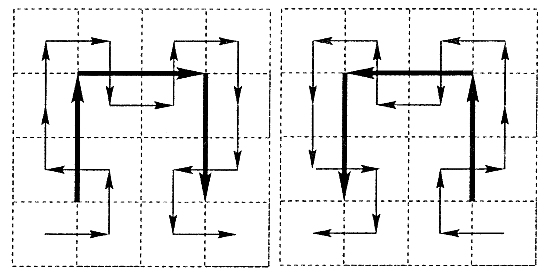

Fig. 6. Hilbert curve of 2nd-order. Scanning paths within a 16×16 block (each cell is a 4×4 block).

IV. Dynamic Programming Solution

Sections III-C and III-D dealt with two alternative ways to combine the DCT and fractal transforms in order to approximate an image block. While it was shown how to arrive at an optimal representation of a block given the image partition, the structure of the problem is such that partition and coding decisions are not independent. To take advantage of the expected average intensity continuity between blocks, as is done in JPEG, the c_{i0} coefficients in (7), are encoded differentially, with respect to the previously encoded block. Furthermore, some blocks in the employed three-level QT image decompositions can have up to three predecessors, and, hence, the rate of a current block exhibits a 1st-order dependency. This situation is illustrated by the trellis structure in Fig. 5. Approximating an image block by the proposed hybrid DCT/fractal code can be thought of as quantizing the block. A particular way of combining these two components, characterized by a specific domain block and an isometry, as well as a specific set of significant DCT coefficients, is called a generalized quantizer q_i. Although the number of these quantizers is large, for simplicity, only two are shown in Fig. 5 per block. Differential encoding of the DC coefficient of each range block makes these quantizers dependent. However, for a given range block, no forward dependency exists since the DC coefficient is always quantized to the nearest integer, regardless of the predecessor. In practice, we preprocess each range block to identify the best quantizers [based on the solution of (11)] per predecessor. This step significantly reduces the number of possible paths in the QT decomposition without causing suboptimality.

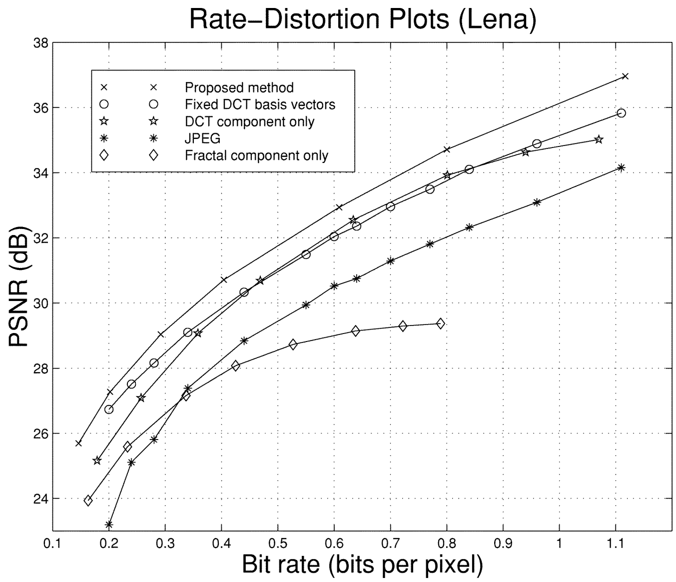

Fig. 7. Rate distortion performance of various techniques (*Lena* image).

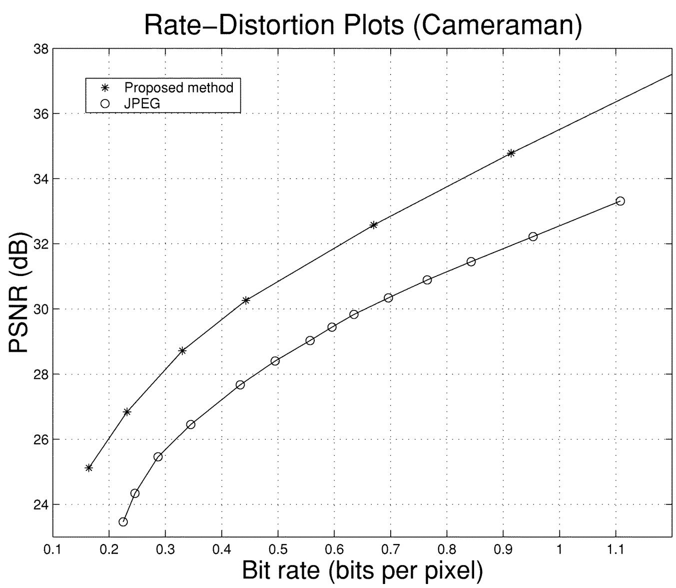

Fig. 8. Rate distortion performance of various techniques (*cameraman* image).

The task of the encoder is to select the image decomposition and the hybrid code for each block such that (5) is satisfied. The Hilbert curve which is known to satisfy certain adjacency requirements [21], fits naturally into the QT decomposition and is efficient for predictive coding. Without lack of generality, in our experiments, we use the 3rd-order Hilbert curve for the 256 × 256 input image. Possible scanning paths within a 16 × 16 block are shown in Fig. 6.

The overall problem of finding the optimal segmentation and the hybrid fractal/DCT code is posed as that of finding the shortest path through the leaves of the quad-tree decomposition, with each leaf having one to three possible codes, corresponding to one to three predecessors of a block in our Hilbert curve. If $g_{i,j} = d_i + \lambda \cdot r_{i,j}$ denotes the transition cost associated with encoding block i with block j as its predecessor, the solution is found as the shortest path in the trellis in Fig. 5, in other words, the problem has the optimal substructure property. This motivates the use of dynamic programming (DP) for finding this solution.

Fig. 9. Proposed algorithm ($R = 0.20$ bpp, $PSNR = 26.71$ dB).

Operational optimality, in our case, is achieved by finding the ordered sequence of generalized quantizers $[q_0^*, \ldots, q_K^*]$, minimizing the overall cost function,

$$J^*(q_0^*, \ldots, q_K^*) = \min_{q_0, \ldots, q_K;\, K} J(q_0, \ldots, q_K) \tag{14}$$

where both the quantizers (generalized codes) and their number K have to be determined by the encoder, with the optimal partition the byproduct of the above minimization.

Due to the fact that the total cost function minimization can be broken down into subproblems in the following recursive manner

$$\begin{aligned} J^*(q_0^*, \ldots, q_K^*) = \min_i \, &[J^*(q_0^*, \ldots, q_i^*) \\ &+ J^*(q_{i+1}^*, \ldots, q_K^*)], \quad 0 \le i \le K. \end{aligned} \tag{15}$$

DP is the natural method to find the shortest path in our QT decomposition [3].

Each value of λ that is kept constant throughout the solution process generates one point in the rate-distortion plane. These points are operationally optimal for the obtained bit rates $(R = R_{\max})$. In order to obtain the RD plots, discussed in Section V, we sweep lambda in some interval $[\lambda_{\min}, \ldots, \lambda_{\max}]$. Alternatively, had a specific operating point been the objective, bi-section or other fast search methods [21] could have been applied to arrive at λ^*.

V. Experimental Results

Fig. 7 compares rate-distortion performance of the proposed hybrid fractal/DCT algorithm, run on a 256 × 256 *Lena* image, with both fixed and optimal DCT subspace selection with that of JPEG, the optimized DCT component only, as

Fig. 10. JPEG ($R = 0.20$ bpp, $PSNR = 23.19$ dB).

Fig. 11. Proposed algorithm ($R = 0.45$ bpp, $PSNR = 31.16$ dB).

Fig. 12. Proposed algorithm: DCT component only ($R = 0.45$ bpp, $PSNR = 30.44$ dB).

Fig. 13. Proposed algorithm: fractal component only ($R = 0.45$ bpp, $PSNR = 28.26$ dB).

well as, the purely fractal approach. The hybrid algorithm with the optimal subspace is shown to outperform all other competing techniques, with JPEG outperformed by 2.5–3.5 dB and DCT component only by 0.5–1.5 dB across the range of bit rates. Similar results were obtained with other images. As an example, Fig. 8 demonstrates performance of the proposed algorithm (with an optimized DCT subspace) and that of JPEG when executed on the *cameraman* image.

The images compressed by JPEG and the proposed algorithm (with the fixed DCT subspace) at roughly the same bit rate (0.20 bpp) are shown in Figs. 9 and 10, respectively. The hybrid algorithm reduces the blocking artifacts by modeling high frequency information through the fractal transform. Fig. 1 shows how the optimal segmentation adapts to the image by using larger size range blocks in relatively uniform areas. These show that the fractal transform is efficient for representing high frequency information, and the DCT for representing low frequency information.

It is interesting to observe how the proposed algorithm (with the optimal subspace selection) compares to its two constituent methods run in isolation. Figs. 11–13 show the decoded images generated by using the hybrid transformation, DCT component only (based on an optimal coefficient selection), and the fractal component only, respectively, at the rate of 0.45 bits per pixel (bpp), corresponding to approximately 18 : 1 compression ratio.

Clearly, adding the orthogonal fractal transform to the optimized DCT component significantly improves its performance. It is noteworthy to observe how a small percentage of fractal

TABLE I
BREAKDOWN OF BITS AMONG FRACTAL, DCT, AND SEGMENTATION COMPONENTS USING THE OPTIMIZED DCT METHOD (*LENA* IMAGE)

Bit rate (bpp)	PSNR (dB)	Fractal bits (%)	DCT only blocks (%)	DCT bits (%)	Segmentation bits (%)
1.26	37.21	17.25	30.56	78.69	4.06
1.01	36.10	14.19	37.88	81.93	3.88
0.80	34.71	11.99	43.90	84.26	3.75
0.61	32.94	10.57	47.13	85.68	3.75
0.45	30.94	8.23	54.26	88.05	3.72
0.23	27.07	9.01	63.84	85.57	5.42
0.17	25.43	8.99	69.52	84.84	5.42

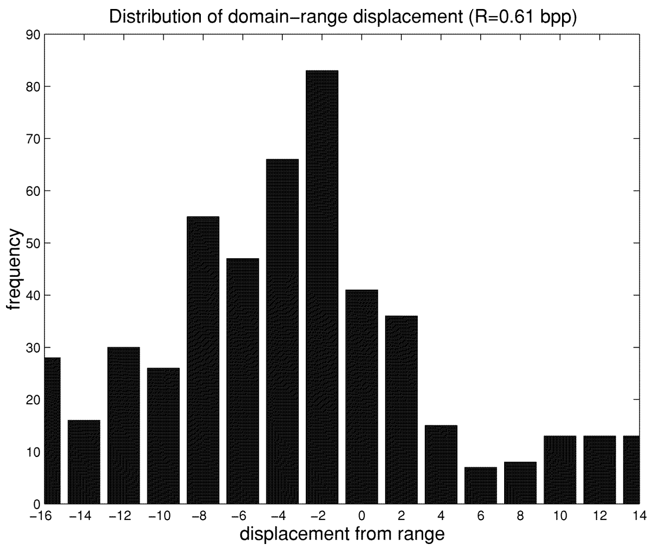

Fig. 14. Distribution of domain-range displacement.

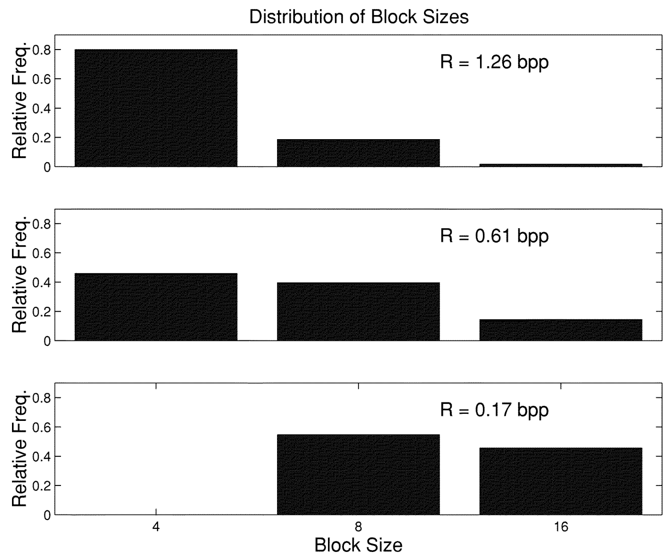

Fig. 15. Distribution of block sizes.

bits can have the said impact on performance. Table I details the breakdown of bits between the fractal and DCT components at various bit rates using the optimized DCT subspace method. It also shows the overall percentage of blocks (DCT only blocks column) in which the fractal component was not used.

Fig. 14 shows the distribution of domain-range displacements in the coded *Lena* image at 0.61 bpp. In general, the variance of the distribution increases with bit rate, which suggests that the use of multiple VLC tables at different rates may lead to further gains [12].

A similar observation can be made with respect to the distribution of block sizes in the coded *Lena* image at the same rates. Fig. 15 illustrates the change in the histogram of coded block sizes with the rate. As expected, at high bit rates, more 4×4 blocks are used. At low bit rates, however, 8×8 and 16×16 blocks dominate the histogram.

Fig. 1 demonstrates a QT partition (overlaying the original) that resulted from the application of the proposed algorithm (optimal DCT subspace) to the *Lena* image with the target rate of 0.45 bpp and the resulting distortion of 30.33 dB. Again, because the hybrid code and the partition were optimized jointly, the encoder adaptively selected larger blocks in relatively uniform areas and smaller blocks in areas with significant high frequency content.

VI. CONCLUSIONS

In this paper we have presented a novel image compression technique based on the hybrid fractal/DCT approximation. Using a QT decomposition, we jointly optimized the image partition, and the fractal and the DCT components of the code. We obtained rate-distortion efficiency from the hybrid coder surpassing that of its constituent components used in isolation.

The derived results confirm the original hypotheses that the combination of DCT and fractal transforms is efficient at capitalizing on both short and long-range correlations present in typical images, and that the two transforms are better suited at decorrelating low and high frequencies, respectively.

It should be noted that the optimality claimed here applies only in the operational sense. The experimental results suggest that hybrid parameter distributions are very sensitive to the operating point (bit rate). A carefully chosen set of VLC tables for various coding parameters is likely to result in Rate Distortion curves superior to those presented in Section V. One possible way to extend the claim of optimality to the VLC tables used is through the use of iterative VLC refinement, as was done in [12].

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