Fractal Image Approximation and Orthogonal Bases

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Abstract

We are concerned with the fractal approximation of multidimensional functions in $\mathcal{L}^2$. In particular, we treat a position-dependent approximation with no search using orthogonal bases of $\mathcal{L}^2$. We describe a framework that establishes a connection between the classic orthogonal approximation and the fractal approximation.

The main theorem allows easy and univocal computation of the parameters of the approximating function. From the computational perspective, we can avoid to solve linear systems often suffering from ill conditioning and needed in former fractal approximation techniques. Moreover, using orthogonal bases we obtain the most compact representation of the approximation. As a direct application we show some results on the compression of gray scale digital images.

Key words: deterministic fractal geometry, approximation theory, image compression, orthogonal bases

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1 Introduction

Some years ago it has been shown that the deterministic fractal geometry is capable to produce complex behaviors even with simple mathematical models [3]. Fractal models appear suitable to represent real world images [6,14,20,21].

In 1987, Barnsley originally proposed the idea to use deterministic fractal geometry to obtain a compressed representation of digital images. Some years

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later, one of his students devised the first algorithm capable to partially achieve that goal [10].

The idea of fractal coding is to represent the signal, or better, the function that one wants to approximate, solely by the relations that are present between affinely transformed parts of the signal and the signal itself. Through the removal of of "self-affine redundancy", one hopes to obtain a more compact representation than the original one.

Barnsley [4], Jacquin [10–12] and Fisher [9] presented different methods for looking for the similarities present in digital images. For simplicity of implementation the search for similarity was performed only between *blocks* in which the image was preventively decomposed. The brightness of a block was being approximated by a linear transformation of the brightness of another bigger block. Among all the bigger candidate blocks it was chosen the one which approximates better the original, together with a particular transformation.

The whole image was hence represented through the relationship between blocks and by the coefficients of such brightness transformation. They originally chose linear transformations with a constant translation term (with respect to the position inside the block). Although later many other strategies has been proposed (see e.g., [1]) the search process was always highly computationally intensive.

A different approach, motivated by the desire to reduce substantially the computational cost, has been proposed by Monro and Dudbridge in which the approximation is applied independently on each single block [16]. The method, although simple to implement and very fast, does not perform well. It constrains too strong auto-similarities inside the blocks, that are generally not present in real word images. To obtain a better quality of the approximation the authors propose to substitute the constant translation term with a polynomial in the pixels coordinates. The polynomial approximates the residual error that cannot be captured by the fractal approximation.

Barnsley himself introduced, in the one-dimensional case, a class of fractal interpolation functions which have a self-similarity property [2]. It is possible to reformulate that theory in terms of fractal approximation functions in $\mathcal{L}^2(\mathbb{R}^n)$.

The work described in this paper should be intended as a way to accomplish such purpose. In particular, since we are going to treat the problem of image coding (i.e., the approximation of a brightness function) we treat, without loss of generality, the two-dimensional case.

We describe a more general type of position dependent approximation than the one by Monro and Dudbridge, in which the translation term is a function that belongs to the subspace generated by a chosen orthogonal basis. In literature other orthogonal techniques used in fractal coding are [19,18].

The main result of this work is a theorem that builds such approximation from an approximation of the gray scale function expressed with respect to the same basis. Since the resulting approximation is optimal with regards to the chosen basis we will call it the best fractal orthogonal approximation (BFOA).

In practise, if you suppose to have a 'classic' place dependent approximation the simple rules of the theorem 'turn it' into a fractal approximation. In this way we can avoid to use heavy numerical methods to overcome the ill conditioned problems associated to the kind of polynomials in [16,17,15].

We will show some results obtained with cosine and Haar basis. We want to emphasize that the initial approximation can be computed with any algorithm, for example with fast technique like FFT or DWT.

Our work proposes a new approximation model, and not yet another compression technique.

Section 2 recall some notations used in the rest of the paper. In section 3 we introduce a theory for fractal approximation in $\mathcal{L}^2(\mathbb{R}^2)$ with a variant of the Collage theorem. In section 4 we present the main result of the BFOA and an issue on the contractivity of the operator.

The application of BFOA to image is described in section 5. We show the use of different orthogonal bases in a block coding framework. Our approach allows to obtain a reconstruction error, at a parity of parameters, lower than the polynomial approximation. Moreover we investigate the best splitting point of the blocks as a searching method and we show the advantages given by the utilization of bigger blocks than the one usually found in literature.

2 Notations

We briefly recall some notation used in the paper. We consider functions in $\mathcal{L}^p$ with the metric

$$d(f,g) = \|f-g\|_p$$

where

$$\|f\|_p = \left(\int_{\mathbb{R}^2} |f(x)|^p \, d\mu \right)^{1/p}$$

If f is a function in $\mathcal{L}^p$, with *best approximation* of f in $\mathcal{L}^p$ we denote the function $f^* \in \mathcal{L}^p$ that satisfies

$$\|f - f^*\|_p = \inf_{g \in \mathcal{L}^p} \|f - g\|_p$$

We chose $p = 2$ because $\mathcal{L}^2$ is rich of properties and because the $\mathcal{L}^2$–norm is the easiest norm to handle. We will denote with $\langle \cdot, \cdot \rangle$ the scalar product, with $\circ$ the operator that composes two functions and with $T^n(f)$ the iterated application of T to f, n times.

3 Fractal approximation in $\mathcal{L}^2(\mathbb{R}^2)$

We identify a continuous gray scale image with a function $f \in \mathcal{L}^2$ which domain is a compact set A, attractor of an IFS $\{A;\ w_1, \ldots, w_N\}$ (see [3])

$$A = \bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} w_i(A)$$

where the maps w_i are affine, contractive and non-overlapping [1] i.e., $w_i(x) = L_i x + \tau_i,\ x \in \mathbb{R}^2$, L_i are 2×2 scaling matrices, and τ_i are translation vectors. The maps w_i describe the underlying "geometry" of the domain A of the function f.

A fractal approximation of f is a function f^* associated with an $\mathcal{L}^2$-contractive operator such that f^* is the fixed point of T, that is $Tf^* = f^*$.

Since the metric space $(\mathcal{L}^2(A), \|\cdot\|_2)$ is complete, by the Banach's theorem there is only one fixed point that can be obtained by the following *reconstruction algorithm*

$$\lim_{n \to \infty} T^n(g) = f^*, \quad \forall g \in \mathcal{L}^2(A) \tag{1}$$

The procedure (1) permits to obtain f^* by the iterations of the operator T starting from *any* initial function g.

The operator T is usually built from the IFS maps w_i and from some appropriate functions F_i

$$(Th)(x) = F_i(h(w_i^{-1}(x))), \quad \forall x \in A_i,\ h \in \mathcal{L}^2(A)$$

[1] $w_j(A) \cap w_k(A) = \emptyset \quad \forall j, k = 1, \ldots, N$ with $j \neq k$.

In a more general case, we can consider a *place-dependent* operator

$$(Th)(x) = F_i(w_i^{-1}(x), h(w_i^{-1}(x))), \quad \forall x \in A_i,\ h \in \mathcal{L}^2(A) \tag{2}$$

where $F_i : A \times [c,d] \to [c,d],\ i = 1, \ldots, N$ are functions satisfying the Lipschitz condition

$$|F_i(x, y_1) - F_i(x, y_2)| \le s_i |y_1 - y_2|, s_i > 0, \quad \forall x \in A,\ \forall y_1, y_2 \in [c,d]$$

When $(\sum_{i=1}^N |\det L_i|\ s_i^2)^{1/2} < 1$ then T is contractive in $\mathcal{L}^2(A)$. Indeed, since the sets A_i are disjoint, we have

$$\begin{aligned}
&\|Th - Tg\|_2^2 = \\
&\sum_{i=1}^N \int_{A_i} \left|F_i(w_i^{-1}(x), h(w_i^{-1}(x))) - F_i(w_i^{-1}(x), g(w_i^{-1}(x)))\right|^2 d\mu = \\
&\sum_{i=1}^N |\det L_i| \int_A |F_i(x, h(x)) - F_i(x, g(x))|^2 d\mu \le \\
&\sum_{i=1}^N |\det L_i|\, s_i^2 \int_A |h(x) - g(x)|^2 d\mu = \\
&\sum_{i=1}^N |\det L_i|\, s_i^2 \|h - g\|_2^2
\end{aligned} \tag{3}$$

Of particular interest is the linear case

$$F_i(x, y) = \alpha_i y + q_i(x), \quad \alpha_i \in \mathbb{R}, \quad q_i \in \mathcal{L}^2(A) \tag{4}$$

in which the operator T becomes

$$(Th)(x) = \alpha_i h(w_i^{-1}(x)) + q_i(w_i^{-1}(x)), \quad \forall x \in A_i,\ h \in \mathcal{L}^2(A) \tag{5}$$

From (3) it follows that if $(\sum_{i=1}^N |\det L_i|\ \alpha_i^2)^{1/2} < 1$ then T is contractive in $\mathcal{L}^2(A)$. It is interesting to remark that T can be contractive even if not all the maps satisfy $|\alpha_i| < 1$.

3.1 The inverse problem

The problem of finding a contractive operator T whose fixed point is f, or better, close to f, is called the *inverse problem*. The Collage theorem provides directions on how to evaluate a given operator.

Theorem 1 *Let $f \in \mathcal{L}^2(A)$, $T : \mathcal{L}^2(A) \to \mathcal{L}^2(A)$ be the contractive operator defined in (2) with contractivity factor $0 \le K < 1$, and f^* its fixed point. If*

$$\|f - Tf\|_2 < \epsilon$$

then

$$\|f^* - f\|_2 < \frac{\epsilon}{1-K}$$

or equivalently,

$$\|f^* - f\|_2 < (1-K)^{-1}\|f - Tf\|_2$$

PROOF. The proof proceeds as for the classic IFS theory [3]. □

The theorem states that once we want to approximate a particular $f \in \mathcal{L}^2(A)$, if T is such that f and its image under T are 'near enough', then f will be 'near enough' to f^*.

Note that theorem is not constructive. It provides a measure of the quality of the approximation without having to calculate the fixed point of T, but it does not suggest any method to find explicitly T.

3.2 Best fractal approximation in $\mathcal{L}^2$

In the rest of the paper we consider the case of the linear operator described in (5). In particular, we assume that the functions q_i have been chosen in a subspace U of $\mathcal{L}^2(A)$.

We call *best fractal approximation* of f the fixed point f^* of the operator T such that f has minimum distance from Tf. That is

$$\inf_{q_i \in U,\ K<1} \|f - (\alpha_i\ f \circ w_i^{-1} + q_i \circ w_i^{-1})\|_2$$

where $K = (\sum_{i=1}^{N} |\det L_i|\ \alpha_i^2)^{1/2}$.

Alternatively, the search for the best fractal approximation of f can be carried out by looking for the parameters α_i and the functions q_i which minimize

$$\|f \circ w_i - (\alpha_i f + q_i)\|_2^2 \quad \forall i = 1, \ldots, N \tag{6}$$

since

$$\begin{aligned}
&\|f - (\alpha_i\ f \circ w_i^{-1} + q_i \circ w_i^{-1})\|_2^2 = \\
&\sum_{i=1}^{N} \int_{A_i} \left| f(x) - (\alpha_i f(w_i^{-1}(x)) + q_i(w_i^{-1}(x))) \right|^2 d\mu = \\
&\sum_{i=1}^{N} |\det L_i| \int_{A} |f(w_i(x)) - (\alpha_i f(x) + q_i(x))|^2\, d\mu = \\
&\sum_{i=1}^{N} |\det L_i|\ \|f \circ w_i - (\alpha_i f + q_i)\|_2^2
\end{aligned}$$

4 The best fractal orthogonal approximation

When the functions q_i belong to the subspace generated by an orthogonal basis, the operator T can be obtained by fairly simple rules.

The following theorem allows to construct the function that approximate $f \circ w_i$, i.e., the function that minimizes $\|f \circ w_i - (\alpha_i f + q_i)\|_2$ with respect to $q_i \in U$, $\alpha_i \in \mathbb{R}$. We call such approximating function the *best fractal orthogonal approximation* of $f \circ w_i$ in U.

Theorem 2 *Let $f \in \mathcal{L}^2(A)$, $A \subset \mathbb{R}^2$ be a compact with $\mu(A) < +\infty$ and $\{w_1, \ldots, w_N\}$ be non-overlapping contractive affine maps, such that A is the attractor of the associated IFS. Let $\{u_0, u_1, \ldots, u_n\}$ denote an orthogonal system in $\mathcal{L}^2(A)$, U the subspace generated by its elements, $\sum_{j=0}^{n} c_j u_j$ and $\sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j$, respectively the best approximation in U of f and $f \circ w_i$. Then, for each $i = 1, \ldots, N$, there is an element in U, $g_i = \sum_{j=0}^{n} \varphi_j^{(i)} u_j$, univocally determinate by*

$$\varphi_j^{(i)} = \tilde{c}_j^{(i)} - \hat{\alpha}_i c_j, \quad \hat{\alpha}_i = \frac{\int_A f \cdot f \circ w_i \, d\mu - \sum_{j=0}^{n} \|u_j\|_2^2 \, c_j \tilde{c}_j^{(i)}}{\int_A f^2 \, d\mu - \sum_{j=0}^{n} \|u_j\|_2^2 \, c_j^2} \tag{7}$$

such that $\hat{\alpha}_i f + g_i$, $i = 1, \ldots, N$ *is the best fractal orthogonal approximation of* $f \circ w_i$ *in* U.

If $\int_A f^2 \, d\mu - \sum_{j=0}^{n} \|u_j\|_2^2 \, c_j^2 = 0$, *we have to set* $\hat{\alpha}_i = 0$; *in this case* g_i *agrees exactly with the best approximation of* $f \circ w_i$ *in* U.

PROOF. First of all, note that if $\sum_{j=0}^{n} c_j u_j$ and $\sum_{j=0}^{n} d_j u_j$ are, respectively, the best approximation of f and g, with respect to the orthogonal basis $\{u_j\}_{j=0,\ldots,n}$, we have

$$\langle f - \sum_{j=0}^{n} c_j u_j, \, g - \sum_{j=0}^{n} d_j u_j \rangle \; = \; \langle f, \, g \rangle - \sum_{j=0}^{n} \|u_j\|_2^2 \, c_j d_j \tag{8}$$

The reason is explained in the following steps

$$\begin{aligned}
&\langle f - \sum_{j=0}^{n} c_j u_j, \, g - \sum_{j=0}^{n} d_j u_j \rangle = \\
&\langle f, \, g \rangle - \sum_{j=0}^{n} d_j \langle f, \, u_j \rangle - \sum_{j=0}^{n} c_j \langle g, \, u_j \rangle + \sum_{j=0}^{n} \sum_{k=0}^{n} c_j d_k \langle u_j, \, u_k \rangle = \\
&\langle f, \, g \rangle - \sum_{j=0}^{n} \|u_j\|_2^2 \, c_j d_j
\end{aligned}$$

where

$$\begin{gathered}
\langle u_j, \, u_k \rangle = 0, \text{ then } k \neq j, \quad \langle u_j, \, u_j \rangle = \|u_j\|_2^2 \; , \\
c_j = (\|u_j\|_2^2)^{-1} \langle f, \, u_j \rangle \; , \quad d_j = (\|u_j\|_2^2)^{-1} \langle g, \, u_j \rangle \; , \\
j = 0, \ldots, n, \quad \|u_0\|_2^2 = \int_A d\mu = \mu(A),
\end{gathered}$$

and where u_0 is the unitary function.

Fixed the coefficients α_i, let $g_i = \sum_{j=0}^{n} \varphi_j^{(i)} u_j$ be the best approximation in U of $f \circ w_i - \alpha_i f$. Therefore, $\varphi_j^{(i)}$ are the Fourier coefficients of $f \circ w_i - \alpha_i f$.

We define now an auxiliary function

$$G(\alpha_i) = \|f \circ w_i - (\alpha_i f + g_i)\|_2^2 = \|(f \circ w_i - \alpha_i f) - g_i\|_2^2 \tag{9}$$

It is well known that

$$G(\alpha_i) = \min_{q_i \in U} \|(f \circ w_i - \alpha_i f) - q_i\|_2^2$$

In addition to this, the minimum is unique. It follows that

$$\varphi_j^{(i)} = (\|u_j\|_2^2)^{-1} \langle f \circ w_i - \alpha_i f,\, u_j \rangle =$$
$$(\|u_j\|_2^2)^{-1} \langle f \circ w_i,\, u_j \rangle - \alpha_i\, (\|u_j\|_2^2)^{-1} \langle f,\, u_j \rangle \;=\; \tilde{c}_j^{(i)} - \alpha_i c_j$$

and hence

$$g_i = \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j - \alpha_i \sum_{j=0}^{n} c_j u_j$$

where

$$c_j = (\|u_j\|_2^2)^{-1} \langle f,\, u_j \rangle, \quad \tilde{c}_j^{(i)} = (\|u_j\|_2^2)^{-1} \langle f \circ w_i,\, u_j \rangle.$$

Substituting in (9) we have

$$G(\alpha_i) = \|(f \circ w_i - \alpha_i f) - (\sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j - \alpha_i \sum_{j=0}^{n} c_j u_j)\|_2^2 =$$
$$\|(f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j) - \alpha_i (f - \sum_{j=0}^{n} c_j u_j)\|_2^2 =$$
$$\int_A [(f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j) - \alpha_i (f - \sum_{j=0}^{n} c_j u_j)]^2 d\mu$$

Now, once we consider G a function of α_i, we can see that it is derivable and it has a unique stationary point given by the solution of

$$\int_A (f - \sum_{j=0}^{n} c_j u_j)\, (f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j)\, d\mu = \alpha_i \int_A (f - \sum_{j=0}^{n} c_j u_j)^2\, d\mu \tag{10}$$

By (8) we have $G^{(2)}(\alpha_i) = 2(\int_A f^2\, d\mu - \sum_{j=0}^{n} \|u_j\|_2^2\, c_j^2)$. Using the Bessel's inequality and the hypothesis of the theorem it follows that $G^{(2)}(\alpha_i) > 0$, that means the point is a minimum. The equation (10) can be rewritten as follows

$$\alpha_i \langle f - \sum_{j=0}^{n} c_j u_j,\, f - \sum_{j=0}^{n} c_j u_j \rangle = \langle f - \sum_{j=0}^{n} c_j u_j,\, f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j \rangle$$

Finally, using (8), we have

$$\alpha_i \left(\langle f,\, f \rangle - \sum_{j=0}^{n} \|u_j\|_2^2\, c_j^2 \right) = \langle f,\, f \circ w_i \rangle - \sum_{j=0}^{n} \|u_j\|_2^2\, c_j \tilde{c}_j^{(i)}$$

and we obtain, as a solution, the value $\hat{\alpha}_i$ in (7). □

4.0.1 Remark

Actually, we do not have any guarantee that using (7) we will satisfy the condition $K < 1$, since the parameters are obtained through an unconstrained minimization. However, we have always verified experimentally the contractivity condition in our test cases with different approximation orders, i.e., choices of n. We can conjecture the general result which formal proof seems pretty difficult. However, under some particular condition, we can show that the operator T obtained by (7) is contractive in $\mathcal{L}^2(A)$.

Proposition 3 *Let $f \in \mathcal{L}^2(A)$ be non-negative function with $\|f\|_2 \neq 0$. When considering the zero-order approximation, i.e. $n = 0$, and $\alpha_i \geq 0, \varphi_0^{(i)} \geq 0$, $\varphi_0^{(i)}$ are not all zero, the operator T whose parameters are obtained by (7) is contracting in $\mathcal{L}^2(A)$.*

PROOF. Rewriting (7) when $j = 0$ we have

$$\alpha_i \left(\|f\|_2^2 - c_0^2 \right) = \langle f,\, f \circ w_i \rangle - c_0\, \tilde{c}_0^{(i)}$$

and hence

$$\alpha_i\, \|f\|_2^2 + \|f\|_1\, \varphi_0^{(i)} = \langle f,\, f \circ w_i \rangle$$

being $\varphi_0^{(i)} = \tilde{c}_o^{(i)} - \alpha_i c_0$, and f non-negative. By the Schwartz's inequality

$$\alpha_i \, \|f\|_2^2 + \|f\|_1 \, \varphi_0^{(i)} \leq \|f\|_2 \cdot \|f \circ w_i\|_2 \tag{11}$$

For convenience of notation, we indicate $D_i = |\det L_i|$. Raising (11) to square, multiplying by D_i and summing over i we get

$$\|f\|_2^4 \sum_{i=1}^{N} D_i \, \alpha_i^2 + \|f\|_1^2 \sum_{i=1}^{N} D_i \, (\varphi_0^{(i)})^2 + 2 \, \|f\|_2^2 \, \|f\|_1 \sum_{i=1}^{N} D_i \, \alpha_i \, \varphi_0^{(i)} \leq \|f\|_2^4$$

since

$$\|f\|_2^2 = \sum_{i=1}^{N} D_i \ \|f \circ w_i\|_2^2$$

Hence

$$\|f\|_2^4 \, (\sum_{i=1}^{N} D_i \, \alpha_i^2 - 1) + \|f\|_1^2 \sum_{i=1}^{N} D_i \, (\varphi_0^{(i)})^2 + 2 \, \|f\|_2^2 \, \|f\|_1 \sum_{i=1}^{N} D_i \, \alpha_i \, \varphi_0^{(i)} \leq 0$$

When $\alpha_i \geq 0, \varphi_0^{(i)} \geq 0$ and $\varphi_0^{(i)}$ are not all zero, the left member is strictly greater than $\|f\|_2^4 \, (\sum_{i=1}^{N} D_i \, \alpha_i^2 - 1)$.

Finally, we have

$$\|f\|_2^4 \, (\sum_{i=1}^{N} D_i \, \alpha_i^2 - 1) < 0$$

that guarantees the required contractivity since $\|f\|_2 \neq 0$ by hypothesis. □

5 Applications to block image coding

In order to discuss the application of the BFOA to image coding we consider the image decomposed in square blocks of 8×8 or 16×16 pixels. A single block becomes the domain A of the brightness function f, the function we want to approximate. We choose w_1, w_2, w_3, w_4 as the functions that map a square in its four equal sub-quadrants. Later we will discuss a more general subdivision of blocks.

The orthogonal system $\{u_i\}_{i=0,\ldots,n}$ we chose first were Legendre and Chebychev polynomials. However, the best results were obtained with the cosine basis

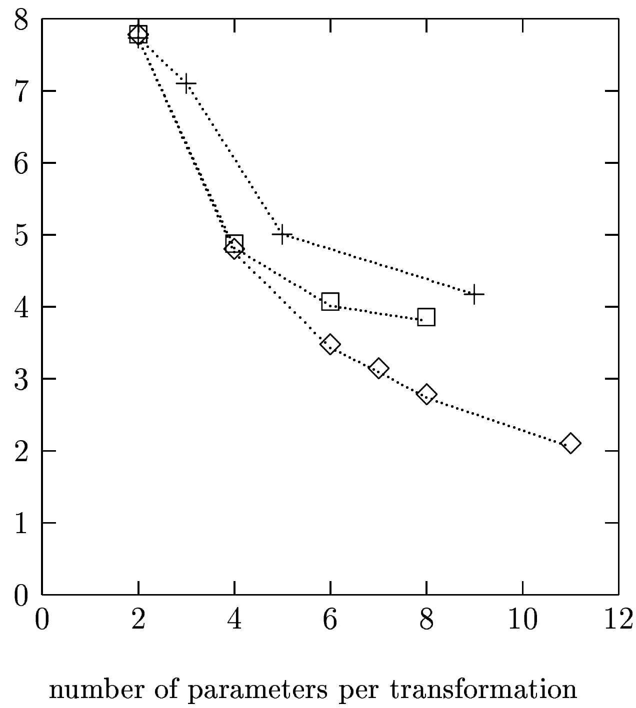

Fig. 1. RMS error for 'Lena' choosing different bases: ◇ cosine basis, + Haar basis, □ Monro-Dudbridge polynomial.

which is briefly described in the appendix. Also we employed a Haar wavelet basis (see appendix).

For each block we compute from (7) the coefficients α_i and $\varphi_j^{(i)},\ i = 1, \ldots, 4$ which build the best fractal orthogonal approximation. The encoding of the block is represented only by these coefficients. The problem of quantization of the encoding is outside the scope of this paper. Our work proposes a new approximation model, and not yet another compression technique.

However, we compare our results with a similar place-dependent method which uses standard polynomials for the q_i, called the Bath Fractal Transform [17,16,15], when no search is performed. In the BFT, the authors obtain the coefficients of the polynomials and α_i by a least square optimization performed by a numerical resolution of linear systems.

In table 1 and in graph 1 we show the approximation error evaluated in the $\mathcal{L}^2$–norm for the image of 'Lena' (Fig. 3) using the BFOA with different orthogonal bases. Our approach gives a reconstruction error, at a parity of parameters, lower than the polynomial approximation in [15]. Moreover, not having ill conditioned problems present in the BFT, it is possible to get an approximation error as low as we want, by simply raising the order.

Finally, in order to evaluate the benefits of a more general partition of the block, we implemented an adaptive searching of the best splitting point of the

	Monro-Dudbridge polynomial	
order	parameters	RMS
0	2	7.73009
1	4	4.81795
2	6	4.01304
3	8	3.80255

		Cosine basis
order	parameters	RMS
0	2	7.73009
1	4	4.75527
*	6	3.42942
2	7	3.09499
*	8	2.73753
3	11	2.05501

		Haar basis
resolution factor	parameters	RMS
0	2	7.73009
1	3	7.10770
2	5	5.00614
3	9	4.17559
4	17	0.00000

Table 1
RMS error for 'Lena'. Blocks 8×8 pixels. With * we indicate a choice of intermediate order. The order for cosine basis is m as in (12). The resolution factor is m as in (13).

IFS describing the domain A. The motivation was to understand how much a better representation, in the fractal sense, of the domain of f could contribute to a better overall approximation.

Let us assume that the block dimension is $P \times P$ pixels. Each choice of (r, s) in the set of admissible points

$$\mathcal{D} = \{(\frac{k_1}{P}, \frac{k_2}{P}) \mid (k_1, k_2) \in [1, \ldots, P-1] \times [1, \ldots, P-1]\},$$

has associated an IFS $\{A, w_1, w_2, w_3, w_4\}$ with attractor A. An example of the choice of the maps w_i in shown in Fig. 2. We consider the best splitting point that one whose IFS minimizes the equation (6).

The optimization problem in $\mathcal{D}$ is solved through a gradient descent method, starting from the center of the block. We verified that our algorithm converges almost surely to the global minimum and it checks on average one fourth of the points in $\mathcal{D}$.

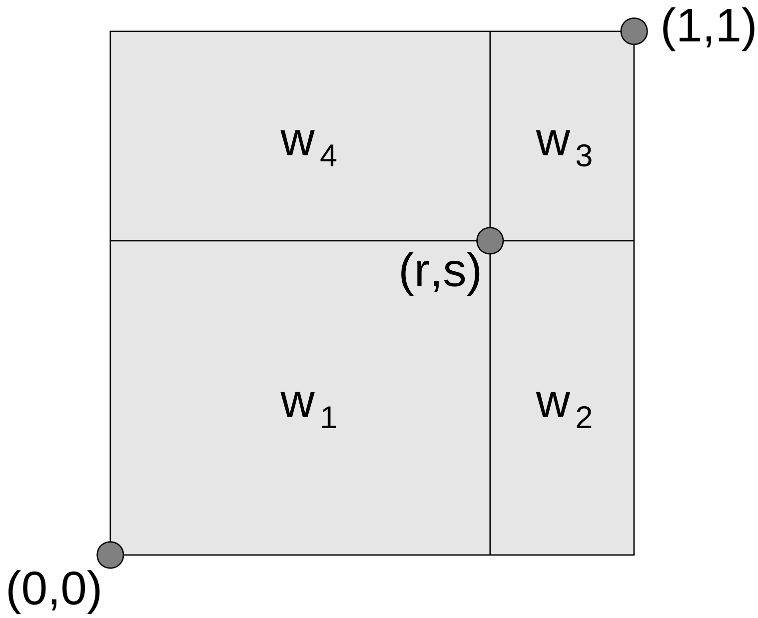

Fig. 2. the subdivision of A in $w_i(A)$.

block size	basis	parameters	data/total parameters	RMS
8×8	Haar	4	4:1	5.36103
8×8	Cosine	4	4:1	4.34129
16×16	Cosine	16	4:1	3.80247

Table 2
RMS error for 'Lena' with searching for the best (r, s).

Table 2 summarizes our experiments. Since we can afford to increase the order of the approximation we can safely have a bigger block dimension. If one chooses, for example, 16×16 blocks with 16 parameters per transformation, we have the same ratio data/parameters as for the 8×8 blocks with 4 parameters per transformation, but the reconstruction error is lower (compare Figures 4 and 5, 6).

6 Conclusions

We introduced a general theory for the position-dependent fractal approximation of functions in $\mathcal{L}^2(\mathbb{R}^n)$, called "best fractal orthogonal approximation", that connects IFS and orthogonal bases. Loosely speaking, our method is capable to "transform" a classic place dependent approximation into a fractal approximation.

Fig. 3. ‘Lena’ digitized 512×512, 8 bit per pixel.

Fig. 4. Blocks 8×8 pixels, cosine basis, 4 parameters per transformation with searching for the best (r, s), RMS $= 4.34129$.

Our approach can be very useful in multidimensional signal processing. In particular, we showed an application to two-dimensional discrete data. Compared with other position-dependent approximation described in literature [16,17,15] it yields better quality of the approximation and less computational efforts.

More adaptive geometry, better methods of searching block similarities and more adaptive functional approximation, seems to be the main goals of the future progresses in fractal image compression.

Recently appeared papers [22,13,7,5] propose to search the similarity relations in the wavelet domain of the images. The results are comparable to state of the art methods for image coding and they are attracting new research interests.

Fig. 5. Blocks 8×8 pixels, Haar basis, 4 parameters per transformation with searching for the best (r, s), RMS $= 5.36103$.

Fig. 6. Blocks 16×16 pixels, cosine basis, 16 parameters per transformation with searching for the best (r, s), RMS $= 3.80247$.

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8 Appendix: remarks on orthogonal bases used

In the following we outline the more interesting basis we used in our experimentations: the cosine basis and the Haar wavelet basis.

8.0.2 Cosine Basis

It is well known that the functions

$$u_i(x)\; u_j(y)\;,\;\; i,j=0,1,\cdots \quad \text{where } u_k(t)=\cos k\pi t,$$

form a complete orthogonal system on the set $I^2 = [0,1] \times [0,1]$.

The decomposition of $f(x,y)$ of order m is given by

$$\sum_{i=0}^{m}\sum_{j=0}^{m-i} a_{ij}\, u_i(x)\, u_j(y) = \sum_{0\le i+j\le m} a_{ij}\, u_i(x)\, u_j(y) \tag{12}$$

where

$$a_{ij} = \frac{1}{h_{ij}} \int_0^1\!\!\int_0^1 f(x,y)\, u_i(x)\, u_j(y)\, dxdy\;,$$

with

$$h_{ij} = \int_0^1\!\!\int_0^1 u_i^2\,(x)\, u_j^2\,(y)\, dxdy = \begin{cases} 1 & i=0=j \\ 1/4 & i,j>0 \\ 1/2 & i=0, \text{ or } j=0,\; i\ne j \end{cases}$$

8.0.3 Haar Basis

The theorem 2 is also remarkable because it allows to use the wavelets. We chose the following orthogonal basis of elementary wavelets — a set of function generated by dilation and translation of single function referred to as the 'mother wavelet' ψ —

$$\psi_{jk}\,(x) = 2^{j/2}\; \psi(B^j\, x - k)\;, \quad x\in\mathbb{R}^2\;,\; j\in\mathbb{Z}\,,\; k\in\mathbb{Z}^2$$

where the mother wavelet is

$$\psi(x) = \begin{cases} 1 & x\in[0,1]\times[0,1/2) \\ -1 & x\in[0,1]\times[1/2,1] \end{cases}$$

and the matrix B, called matrix dilation, is

$$\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

This two-dimensional orthogonal basis of wavelets can be considered a generalization in $\mathcal{L}^2(\mathbb{R}^2)$ of the Haar's system [8].

If one chooses the orthogonal basis ψ_{jk}, the decomposition at the resolution factor m is given by

$$\sum_{j=0}^{m} \sum_{k \in \mathbb{Z}^2} a_{jk} \, \psi_{jk}(x) \, , \quad x \in I^2 \tag{13}$$

where the coefficients a_{jk} are

$$a_{jk} = \int_{I^2} f(x) \, \psi_{jk}(x) \, dx$$

The sum in (13) is taken over multi-index k such that $x \in I^2$.

Note that the integrals involved in the computation of the coefficients, with respect to the bases above described, can be easily implemented as summation of brightness. Moreover it is possible to compute directly the coefficients by using efficient algorithms like the FFT for the cosine basis, or the DWT for wavelets.

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