STATISTICAL EVALUATION OF FRACTAL CODING SCHEMES

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ABSTRACT

This paper reports on investigations concerning the convergence of the reconstruction process in fractal coding schemes from a statistical point of view. For a rather general family of "fractal operators" a necessary and sufficient condition for the convergence based on the spectral radius of the fractal operator is provided. Emerging from this condition the probability density function (pdf) of the magnitude of the eigenvalues is formulated which enables to determine a probabilistic measure for the convergence of the reconstruction process. Since the pdf considerably depends on the structure of the operators, various coding schemes can be analyzed with respect to their convergence properties in a statistical sense. The presented results indicate that certain types of operators are less suited for applications in the field of fractal coding compared to others.

1. INTRODUCTION

Coding schemes are termed *fractal* if the input signal is approximated by a unique *fixed point* of a certain class of transformations [1, 2]. Common to all schemes of this type is that not the transformed signal is stored or transmitted, but the transformation instead whose fixed point is demanded to be close to the original signal. The second characteristic of fractal schemes is that the decoder is able to reconstruct the fixed point solely from the knowledge of the transformation parameters. A coding gain can therefore be achieved if the rate of the transformation parameters is significantly smaller compared to the rate of the signal itself.

This paper deals with the analysis of fractal coding schemes especially with the question of convergence at the decoder from a statistical point of view. Initial results have already been published in [3]. The iterative process of reconstructing the fixed point from the fractal code demands all eigenvalues of the fractal operator to lie within the unit circle [4, 5, 6]. These eigenvalues are regarded as realizations of a random process whose characteristics depend on the structure of the fractal operator. This way the convergence of the reconstruction process for various fractal coding schemes can be assessed in a statistical sense.

The paper is organized as follows: Section 2 introduces the mathematical background of the fractal coding and decoding process. A family of linear operators together with the calculation of their corresponding eigenvalues is introduced. The statistical analysis of the convergence properties for this family is treated in section 3. The results in section 4 together with a prospect on future directions of investigations conclude this paper.

2. MATHEMATICAL BACKGROUND

One advantage of fractal coding schemes in contrast to common ones, e.g. DCT based schemes, is that additionally to the bindings between adjacent samples also some sort of *long range correlations* within the signal are exploited. In the context of fractal coding these correlations are termed (partial) self-similarities which arises from the fact that many parts of (natural) signals may be found again in a scaled and/or geometric transformed version within the same signal.

2.1 Encoding process

Encoding of a given signal $\mathbf{x}$ consists in determining the parameters of a mapping equation

$$W(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}, \tag{1}$$

such that

- $W(\mathbf{x})$ has a unique fixed point $\mathbf{x}_f = \mathbf{A}\mathbf{x}_f + \mathbf{b}$,
- the distance $d(\mathbf{x}, \mathbf{x}_f)$ between the original signal $\mathbf{x}$ and the fixed point $\mathbf{x}_f$ is as small as possible and
- the description of the fractal code $(\mathbf{A}, \mathbf{b})$ is as simple as possible.

For the sake of simplicity the *collage theorem* motivates to minimize the distance $d(\mathbf{x}, \hat{\mathbf{x}})$ between the original signal $\mathbf{x}$ and a first-order approximation $\hat{\mathbf{x}}$ of the fixed point rather than the fixed-point itself [7, 4, 8]. The mapping rule (1) consists of a linear part represented by the operator $\mathbf{A}$ and a non-linear offset $\mathbf{b}$ which together serve as fractal code. Under the condition of a convergent reconstruction the fractal code $(\mathbf{A}, \mathbf{b})$ is a unique description for the fixed point $\mathbf{x}_f$.

2.2 Decoding process

The aim of the decoding process is to reconstruct the fixed point $\mathbf{x}_f$ from the fractal code $(\mathbf{A}, \mathbf{b})$. Direct methods simply apply a matrix inversion so that the fixed point is obtained by $\mathbf{x}_f = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$. The only restriction for the algorithm is that $\det(\mathbf{I} - \mathbf{A}) \neq 0$ holds. For large systems, as is the case in most coding applications, iterative methods are more appropriate for this task. Due to *Banach's fixed point theorem* the sequence $\{\mathbf{x}_k\}$ of iterates $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{b}$ converges for any arbitrary initial signal $\mathbf{x}_0$ to the unique fixed point of the mapping equation (1). In contrast to the less restrictive criterion for the direct method, the necessary and sufficient condition for convergence in the iterative case is that the spectral radius $r_\sigma(\mathbf{A})$ which is the largest eigenvalue of the linear part $\mathbf{A}$ is smaller than one [6, 4].

2.3 Family of linear operators

The investigation of the convergence involves the determination of the eigenvalues of the linear operator $\mathbf{A}$. This is almost impracticable for very general matrices. Therefore we restrict ourselves to a rather simple but nevertheless universal family of linear operators $\mathcal{F}(\mathbf{A})$ which is introduced below.

The entire signal $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ with n pixels is segmented into $N_R = n/n_R$ non-overlapping blocks with n_R pixels, called *range blocks*. Each m neighbored blocks form a *range block cluster*. For every range block $\mathbf{x}_{ki}$ with k denoting the cluster index and i the block number within this cluster another block within the signal, called *domain block*, is searched such that the distance between the range- and a transformed version of the domain block is minimized. The transformations mainly consists of a spatial reduction as well as a gray-level scaling and offset.

Though the domain blocks may have arbitrary size and location within the signal an efficient representation of the fractal code demands some restrictions. A very interesting case is obtained if the domain blocks do not overlap and are of the same size as the range block cluster. If additionally each range block within a cluster is approximated by the same domain block then the number of domain blocks is $N_D = N_R/m$ and their size is $n_D = m n_R$. The structure of a typical operator $\mathbf{A}$ is then as follows:

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{A}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ \mathbf{A}_k & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{A}_{N_D} & \mathbf{0} \end{pmatrix} \qquad (2)$$

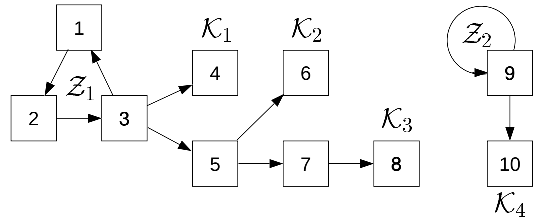

Figure 1 Typical mapping graph for a linear operator $\mathbf{A}$ consisting of two independent mapping cycles $\mathcal{Z}_1 = \{3, 2, 1\}$ of length $L_1 = 3$ and $\mathcal{Z}_2 = \{9\}$ with length $L_2 = 1$ as well as four mapping chains $\mathcal{K}_1 = \{4\}$, $\mathcal{K}_2 = \{6, 5\}$, $\mathcal{K}_3 = \{8, 7, 5\}$ and $\mathcal{K}_4 = \{10\}$ with length $1, 2, 3$ and 1, respectively

Each submatrix

$$\mathbf{A}_k = \begin{pmatrix} \alpha_{ki} \begin{pmatrix} \mathbf{d} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{d} \end{pmatrix} \\ \alpha_{km} \begin{pmatrix} \mathbf{d} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{d} \end{pmatrix} \end{pmatrix} \qquad (3)$$

describes the mapping from one domain block onto the k-th range block cluster. The index $j(k)$ of the selected domain block together with the index k of the range block cluster determine the position of the submatrix $\mathbf{A}_k$ within the operator $\mathbf{A}$. The decimation vector $\mathbf{d}$ is chosen to be $\mathbf{d} = (1, 1, \ldots, 1)/m \in \mathbb{R}^m$, though other choices are possible. The family $\mathcal{F}(\mathbf{A})$ then is the set of all operators $\mathbf{A}$ with the structure as indicated in (2) and (3).

2.4 Mapping graph of the linear operator A

A useful tool to analyze the structure of the operator $\mathbf{A}$ is a *mapping graph* as shown in figure 1. The numbering of the nodes corresponds to the index of the domain blocks or range block clusters within the signal. The edges symbolize the mapping from one domain block onto one range block cluster. Two distinct types of blocks can be distinguished. The first category contains all those blocks which are members of a so called *mapping cycle* indicated by $\mathcal{Z}$. All other blocks in this type of operator belong to open *mapping chains* denoted by $\mathcal{K}$. By introducing the notation

$$j^{\circ p}(k) = \overbrace{j(j(\cdots(j(k)\cdots)))}^{p \text{ times}} \text{ with } j^{\circ 0}(k) = k, k > 0 \qquad (4)$$

and $j(k)$ denoting that the domain block with index $j(k)$ is mapped onto the range block cluster with index k, a mapping cycle can be defined as follows:

Def. 1 The set $\mathcal{Z} = \{k, j(k), j^{\circ 2}(k), \ldots, j^{\circ (L-1)}(k)\}$ is called *mapping cycle* of length L, if L is the least number greater than 0 such that $j^{\circ L}(k) = k$.

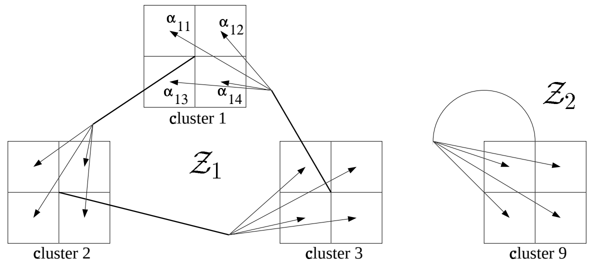

Figure 2 Cycles of operator $\hat{\mathbf{A}}$ with parameter $m = 4$

Convergence only depends on those mappings which are within a mapping cycle. Therefore the operator $\mathbf{A}$ may be reduced to its cyclic part by removing those rows and columns which only describe the blocks of the mapping chains $\mathcal{K}$ [5, 6, 4]. The resulting *cyclic operator* whose mapping cycles are shown in fig. 2 is denoted by $\hat{\mathbf{A}}$. Each of these cycles can be treated independently from all others. This is due to the fact that those part of the signal corresponding to one cycle and parts corresponding to all other cycles are disjointed. For a more detailed description of the mapping the reader is referred to [9].

2.5 Eigenvalues of the linear part

From topic 2.2 it is known, that analyzing the convergence leads to the question whether the eigenvalues of the cyclic operator $\hat{\mathbf{A}}$ lie within the unit circle or not. For the considered family $\mathcal{F}(\mathbf{A})$ of operators a necessary and sufficient criterion for convergence has been published in [6]. Let the index z denote one of Z mapping cycles, L_z the length of this cycle and $\alpha_{li}^{(z)}$ the scaling coefficient corresponding to the mapping onto the i-th range block within the l-th cluster of this cycle. Then convergence is ensured if the spectral radius

$$r_\sigma\left(\hat{\mathbf{A}}\right) = \max_{z\in\{1,2,\ldots,Z\}} \frac{1}{m}\left[\prod_{l=1}^{L_z}\left|\sum_{i=1}^{m}\alpha_{li}^{(z)}\right|\right]^{1/L_z} \qquad (5)$$

of the matrix $\hat{\mathbf{A}}$ is smaller than one.

3. STATISTICAL ANALYSIS

Contractivity is always ensured, if the magnitude of all scaling coefficients α_{li} is strictly smaller than one [1] (this coincides with the l_1-norm criterion). As reported by several authors, e.g. [8, 10], a less stringent restriction for the scaling coefficients improves reconstruction quality as well as convergence speed. On the other hand contractivity of the transformation W is no longer ensured. The aim of this work is therefore to determine the probability for divergence depending on the choice of some design parameters.

Our investigations have shown that the scaling coefficients may be regarded as statistically independent and in $[-\alpha_{\max};\alpha_{\max}]$ uniformly distributed random variables. The spectral radius $r_\sigma\left(\hat{\mathbf{A}}\right)$ as indicator for convergence is solely determined by the scaling coefficients, the length of the mapping cycles L, and the ratio between the domain- and range block size $m = n_D/n_R$ (see eq. 5). Therewith the probability density function (pdf) of the eigenvalues for various choices of the design parameters L and m can be derived. From this pdf the probability for divergent reconstruction can be determined and so the influence of various design parameters on the contractivity can be quantified. In this paper two important special cases of the design parameters are considered which are related to two different fractal coding schemes.

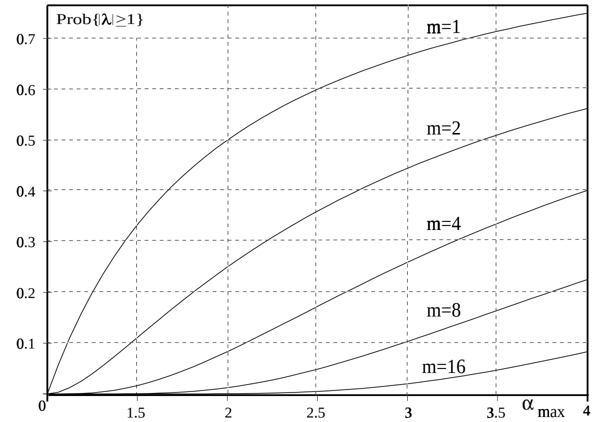

Figure 3 Probability for divergent reconstruction of coding schemes with cycle length $L = 1$

3.1 Case $L_z = 1$

In the first case the mapping function $j(k)$ is defined by $j(k) = k\ \forall k \in \{1,\ldots,N_D\}$ so that each domain block is mapped onto the underlying range block cluster. Therefore the length of all $Z = N_D$ mapping cycles $L_z, z \in \{1,\ldots,Z\}$ equals one. A well known coding scheme with these properties has been published in [11]. Equation (5) can then be simplified so that the largest eigenvalue for one cycle is determined by

$$|\lambda(\mathbf{A})|_{\text{cycle=z}} = \frac{1}{m}\left|\sum_{i=1}^{m}\alpha_{1i}^{(z)}\right|. \qquad (6)$$

As presumed above, the scaling coefficients are uniformly distributed in $[-\alpha_{\max};\alpha_{\max}]$ and also statistically independent. Then the pdf $p_\lambda(\xi)$ of the eigenvalues is equal to the m-fold convolution of the pdf $p_\alpha(\xi)$ of the scaling parameters. The probability for divergence equals the probability that the largest eigenvalue lies outside the unit circle and can easily be determined by

$$\text{Prob}\{|\lambda| \geq 1\} = 2\int_1^\infty p_\lambda(\xi)d\xi. \qquad (7)$$

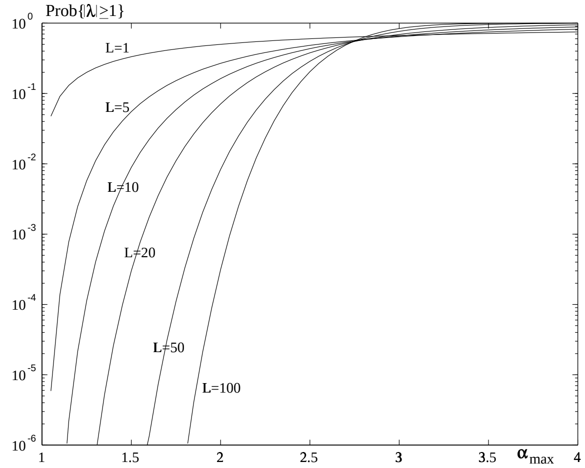

Figure 4 Probability for divergent reconstruction of coding scheme with parameter $m = 1$

Figure 3 shows this probability for some different parameters m as function of the largest allowed scaling coefficient $\alpha_{\max}$. As far as the convergence property is concerned one can see that the choice $m = 1$ is disadvantageous in comparison to larger values for m.

3.2 Case $m = 1$

In the second case the influence of the spatial contraction is excluded so that a range block cluster consists of only one range block with the same size as the domain blocks. This results in the design parameter m being equal to one, so the largest eigenvalue for this type is determined by

$$|\lambda(\mathbf{A})|_{\text{cycle=z}} = \left(\prod_{l=1}^{L_z} \left|\alpha_{l1}^{(z)}\right|\right)^{1/L_z} . \qquad (8)$$

In order to determine the probability of divergence $\text{Prob}\{|\lambda| \geq 1\}$ the pdf $p_{|\lambda|}(\xi)$ has to be calculated which is the pdf of the product of L_z uniformly distributed statistically independent random variables. This results in

$$\text{Prob}\{|\lambda| \geq 1\} = 1 - \frac{1}{\alpha_{\max}^{L_z}} \sum_{l=0}^{L_z-1} \frac{L_z^l}{l\,!} (\ln \alpha_{\max})^l . \qquad (9)$$

As can be seen from figure 4 long mapping cycles are advantageous for a convergent reconstruction process.

4. RESULTS AND CONCLUSION

Summarizing and combining the results of topic 3.1 and 3.2 it can be stated that a fractal coding scheme which is optimized with respect to the contractivity of the transformation should employ

- a large ratio between the domain- and range block size ($m > 1$) and should
- try to generate long mapping cycles ($L \gg 1$).

A coding scheme of the former described type has been implemented and tested with real world images and various choices of decimating factors m. Analyzing the resulting length of the mapping cycles showed that only very short cycles are generated if the selection of the codebook entries is not constrained in any way. By forcing the encoder to generate longer cycles a significant improvement concerning the convergence properties can be obtained.

5. REFERENCES

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