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Contractivity of fractal transforms for image coding

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In this letter the contractivity of existing *fractal transforms* for use in image compression schemes is examined. The coding process is described as non-linear transformation in the finite dimensional euclidean vector space. We derive sufficient conditions for contractivity based on the spectral norm and the spectral radius of the transformation matrix. As a result bounds for the encoding parameters can be formulated which are tighter than the ones known so far.

Introduction: Barnsley's idea to exploit self-similar structures in real world images [1] for compression purposes found its first practical implementation capable of encoding grey-scale images in Jacquin's approach [2]. Several improvements and modifications, e.g. [3, 4, 5, 6, 7] have been reported, where the basic concept of blockwise approximation of the image by parts of itself is adopted.

In this letter we restrict out considerations to the contractivity of the used transformation which is a vital presupposition for the functionality of these schemes. The results of our investigations provide us more freedom in the choice of the encoding parameters which leads to distinct improvements in terms of convergence speed, reconstruction quality and compression ratio.

Theory: Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be an image of size $n = n_x n_y$ pixels which we consider as point in the n-dimensional euclidean vector space $\mathbb{R}^n$. The components $x_i;\ 1 \leq i \leq n;\ x_i \in \mathbb{R}$ represent the pixels of the image. By defining the *Euclidean norm*

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^{n} |x_i|^2} \tag{1}$$

and inducing a *metric*

$$\varrho(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|,\ \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{2}$$

$\mathbb{R}^n$ becomes a *normed metric space* denoted by $(\mathbb{R}^n, \varrho)$. Transformations within this space are described by linear operators $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for which the *spectral- or Hilbert-norm* defined by

$$\|\mathbf{A}\|_{sp} := \sup_{\lambda \in \sigma(\mathbf{A}^T \mathbf{A})} \sqrt{|\lambda|} \tag{3}$$

is a consistent *operator norm* in the sense that $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\|\,\|\mathbf{x}\|$ holds. $\sigma\big(\mathbf{A}^T \mathbf{A}\big)$ is called the *spectrum* of the matrix $\mathbf{A}^T \mathbf{A}$ which is the set of its eigenvalues λ. Additionally for every linear operator $\mathbf{A}$ the *spectral radius* $r_\sigma(\mathbf{A})$ is defined by

$$r_\sigma(\mathbf{A}) := \sup_{\lambda \in \sigma(\mathbf{A})} |\lambda| \leq \|\mathbf{A}\| \tag{4}$$

which is a lower bound for any norm $\|\mathbf{A}\|$.

Most implementations emerge from a blockwise defined non-linear affine transformation

$$W : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \Rightarrow \quad \mathbf{x} \rightarrow \mathbf{A}\mathbf{x} + \mathbf{b} \tag{5}$$

of the entire image $\mathbf{x}$ consisting of a linear part $\mathbf{A}\mathbf{x}$ and an additive part $\mathbf{b}$. The encoding process of the given image $\mathbf{x}$ now consists in finding a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that the approximation error

$$\varrho(W(\mathbf{x}), \mathbf{x}) = \varrho(\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{x}) \tag{6}$$

becomes as small as possible. Data compression can be achieved, if $\mathbf{A}$ and $\mathbf{b}$ can be stored more efficiently than the image $\mathbf{x}$ itself.

Banach's fixed point theorem gives us an idea how the decoding process works:

Let $\mathbb{R}^n$ *be a metric space with metric* ϱ *and* $W : \mathbb{R}^n \to \mathbb{R}^n$ *a contractive transformation, this is if there exists a constant* $s < 1$*, for which*

$$\varrho\left(W(\mathbf{x}), W(\mathbf{y})\right) \leq s\ \varrho(\mathbf{x}, \mathbf{y})\ \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{7}$$

holds. Then the sequence of images $\{\mathbf{x}_k\}$ *constructed by* $\mathbf{x}_{k+1} = W(\mathbf{x}_k)$ *converges for any arbitrary initial image* $\mathbf{x}_0 \in \mathbb{R}^n$ *to the unique fixed point*

$$\mathbf{x}_f = W\left(\mathbf{x}_f\right) = \mathbf{A}\mathbf{x}_f + \mathbf{b};\ \mathbf{x}_f \in \mathbb{R}^n \tag{8}$$

of the transformation W.

From (5), (7) and the definition of the metric (2) we obtain the contractivity condition

$$\|\mathbf{A}\| \leq s < 1, \tag{9}$$

which constrains the choice of the matrix $\mathbf{A}$ in the encoding process.

The decoder generates the sequence $\{\mathbf{x}_k\}$ with

$$\mathbf{x}_k = W^{\circ k}(\mathbf{x}_0) = \mathbf{A}^k\mathbf{x}_0 + \left(\sum_{i=0}^{k-1} \mathbf{A}^{\mathbf{i}}\right)\mathbf{b} \tag{10}$$

by iteratively applying the transformation W (5) to any arbitrary initial image $\mathbf{x}_0$. If the spectral radius satisfies $r_\sigma(\mathbf{A}) < 1$, then $\lim\limits_{k\to\infty} \mathbf{A}^k = \mathbf{0}$ and $\lim\limits_{k\to\infty} \sum\limits_{i=0}^{k} \mathbf{A}^i = (\mathbf{I} - \mathbf{A})^{-1}$ with the identity $\mathbf{I}$ and the null matrix $\mathbf{0}$ [8]. Due to *Banach's fixed point theorem*, the sequence $\{\mathbf{x}_k\}$ converges to the fixed point $\mathbf{x}_f$ of the transformation

$$\lim_{k\to\infty} \mathbf{x}_k = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \mathbf{x}_f\ . \tag{11}$$

An estimation for the reconstruction error can be formulated with the definition of the fixed point (8) and the encoding instruction (6) as

$$\varrho\left(\mathbf{x}_f, x\right) \leq \frac{1}{1 - \|\mathbf{A}\|}\varrho(W(\mathbf{x}), \mathbf{x})\ . \tag{12}$$

Since the spectral radius $r_\sigma(\mathbf{A})$ is a lower bound for every norm $\|\mathbf{A}\|$, the contractivity condition (9) is sufficient but not necessary for convergence. In the following section we investigate the suitability of these results by considering a practical implementation.

Applications: A simple proposal for encoding of single image blocks has been published by Monro and Dudbridge [4]. In the simplest case the encoding process maps an entire image block onto its four underlying subblocks. For the sake of simplicity we treat blocks of size 4x4 pixels. However the theory applies to any other even block size. Then this mapping can be described by a linear operator

$$\mathbf{A} = \frac{1}{4}\begin{pmatrix}
\alpha & \alpha & \alpha & \alpha & 0 & \cdots & & & & & & & & & \cdots & 0 \\
0 & & & & \alpha & \alpha & \alpha & \alpha & & & & & & & & \vdots \\
\vdots & & & & & & & & \alpha & \alpha & \alpha & \alpha & & & & 0 \\
0 & \cdots & & & & & & & & & \cdots & 0 & \alpha & \alpha & \alpha & \alpha \\
\beta & \beta & \beta & \beta & & & & & & & & & & & & \\
0 & & & & \beta & \beta & \beta & \beta & & & & & & & & \\
\vdots & & & & & & & & \beta & \beta & \beta & \beta & & & & \\
0 & & & & & & & & & & & & \beta & \beta & \beta & \beta \\
\gamma & \gamma & \gamma & \gamma & & & & & & & & & & & & \\
0 & \cdots & & & \gamma & \gamma & \gamma & \gamma & & & & & & & & \\
\vdots & & & & & & & & \gamma & \gamma & \gamma & \gamma & & & & \\
0 & & & & & & & & & & & & \gamma & \gamma & \gamma & \gamma \\
\delta & \delta & \delta & \delta & & & & & & & & & & & & \\
0 & & & & \delta & \delta & \delta & \delta & & & & & & & & \\
\vdots & & & & & & & & \delta & \delta & \delta & \delta & & & & \\
0 & \cdots & & & & & & & & & \cdots & 0 & \delta & \delta & \delta & \delta
\end{pmatrix}$$

with the scaling parameters α, β, γ and δ.

We have two possibilities to evaluate these parameters in order to ensure convergence of the transformation. The first one is to determine the norm of $\mathbf{A}$ by calculating the eigenvalues of $\mathbf{A}^T\mathbf{A}$. This matrix is of block-diagonal type with four equal quadratic submatrices. The elements of the submatrices are all identical and equal $\frac{1}{16}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$. The largest eigenvalue which is different from zero is $\frac{1}{4}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ and with the definition of the norm (3) we obtain the contractivity condition

$$\|\mathbf{A}\|_{sp} = \frac{1}{2}\sqrt{(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)} \overset{!}{<} 1 \ . \tag{14}$$

The second way is to determine the spectral radius $r_\sigma(\mathbf{A})$ by directly evaluating the largest eigenvalue of the transformation matrix $\mathbf{A}$. This is in general not feasible but in the special case considered here it is very simple, since the matrix $\mathbf{A}$ has a constant column sum. Therefore the largest eigenvalue in magnitude is the sum of elements in one column and with definition (4) the contractivity condition from the spectral radius is

$$r_\sigma(\mathbf{A}) = \frac{1}{4}|\alpha + \beta + \gamma + \delta| \overset{!}{<} 1 \ . \tag{15}$$

Results and Conclusion: Former results [2, 4] constrained the magnitude of the scaling parameters α, β, γ and δ to be strictly smaller than one in magnitude in order to obtain a contractive transformation W. The results presented here show, that this constraint is to strong and so the relaxed ones (14) or (15) have been derived. The constraint obtained from the spectral radius (15) even offers a more general solution than the one obtained by the spectral-norm (14), since the scaling parameters are positive or negative with same probability and therefore large parameters not necessarily lead to divergent behavior of the transformation.

Enlarging the allowed parameter range consecutively leads to an improvement in terms of reconstruction quality which has already been reported experimentally by other authors [3, 6]. Employing the proposed constraints for the scaling parameters, contractivity of the transformation can be ensured while the parameter range has been significantly enlarged.

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