ON THE PROBLEM OF CONVERGENCE IN FRACTAL CODING SCHEMES

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ABSTRACT

Most fractal coding schemes employ an iterative decoding algorithm in order to reconstruct the approximation of the original signal from the fractal code. A necessary condition for obtaining a unique solution is the convergence of the reconstruction process. This paper reports on investigations concerning a necessary and sufficient condition for convergence which is based upon the spectral radius of the transformation matrix. For a very general class of fractal transforms a simple calculation of the spectral radius can be performed in order to decide whether the reconstruction converges or not. This allows more freedom in the choice of the encoding parameters resulting in a better and faster reconstruction process. Also the proposed description leads to a more accurate theoretical foundation of fractal coding schemes.

1. INTRODUCTION

During the last years a novel coding and modeling scheme for natural signals has been developed which is widely known as *fractal coding*. The basic idea for encoding of signals by use of fractal techniques is originated in various publications of Barnsley et al., e.g. [1]. A first implementation for automatic encoding of gray-scale images at common compression ratios has been proposed by Jacquin [2] and reviewed in [3]. An excellent mathematical foundation of fractal signal modeling based on the theory of finite-dimensional vector spaces is included in [4]. Recently some improvements and modifications have been published, e.g. [5, 6, 7], which make the fractal technique a challenging candidate especially for encoding of images.

In contrast to common transformations, e.g. the DCT, whose coding gain emerges from the bindings between adjacent samples, fractal coding schemes mainly exploit some sort of *long range correlations* within the signal. In the context of fractal coding these correlations are termed (partial) self-similarities which arises from the fact that many parts of natural signals may be found in a scaled and/or geometric transformed version within the same signal. All these schemes can be described as vector quantization with *signal dependent codebook*. The main advantage compared to conventional VQ-algorithms is the fact that no codebook training is necessary and that the vectors may be of very high dimensionality

The encoding process works as follows: The entire signal $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ consisting of n pixels is segmented into $N_R = n/n_R$ non-overlapping blocks $\mathbf{x}_i$ with n_R pixels each. Then for every block $\mathbf{x}_i,\ i = 1, 2, \ldots, N_R$ a codebook entry $\mathbf{y}_j$ from a set of N_D entries is selected, which after scaling with α_{ij} and adding an offset $b_i\mathbf{1}$ minimizes some predefined distortion measure

$$d(\mathbf{x}_i, \hat{\mathbf{x}}_i) = \min_{j \in \{1,2,\ldots,N_D\}} d(\mathbf{x}_i, \alpha_{ij}\mathbf{y}_j + b_i\mathbf{1}) \ . \tag{1}$$

The codebook is generated from the entire image $\mathbf{x}$ by use of a codebook construction matrix $\mathbf{C}$ which is mainly determined by the type of the coding scheme. Let $\mathbf{Cx}$ denote the collection of all codebook entries $\mathbf{y}_j$ and let $\mathbf{F}_j$ be the 'fetch-operation' of the codebook entry $\mathbf{y}_j = \mathbf{F}_j\mathbf{Cx}$ from the codebook. Further let $\mathbf{P}_i$ denote the 'put-operation' of the optimal modified codebook entry $\hat{\mathbf{x}}_i = \alpha_{ij}\mathbf{y}_j + b_i\mathbf{1};\ \mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^{N_R}$ into its appropriate position of the approximated signal $\hat{\mathbf{x}}$. Then the mapping process of the entire signal $W : \mathbb{R}^n \rightarrow \mathbb{R}^n$ may be formulated by

$$\begin{aligned} \hat{\mathbf{x}} &= \sum_{i=1}^{N_R} \mathbf{P}_i(\alpha_{ij}\mathbf{F}_j\mathbf{Cx} + b_i\mathbf{1}) \\ &= \left\{ \sum_{i=1}^{N_R} \mathbf{P}_i\alpha_{ij}\mathbf{F}_j\mathbf{C} \right\} \mathbf{x} + \sum_{i=1}^{N_R} \mathbf{P}_i b_i \mathbf{1} \\ &= \mathbf{Ax} + \mathbf{b} \ . \end{aligned} \tag{2}$$

From (2) it can be seen that this mapping is an affine transformation with a linear part $\mathbf{A}$ and a non-linear offset $\mathbf{b}$. A very simple coding scheme can be stated, if the mapping (2) results in a convergent series of iterates $\mathbf{x}_{k+1} = \mathbf{Ax}_k + \mathbf{b}$, which due to *Banach's fixed point theorem* then converges for any arbitrary initial signal $\mathbf{x}_0$ to the unique *fixed point* $\mathbf{x}_f = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$. A necessary and sufficient condition is that the spectral radius $r_\sigma(\mathbf{A}) = \sup_{\lambda \in \sigma(\mathbf{A})} |\lambda|$, which is the largest eigenvalue of the linear part $\mathbf{A}$ with $\sigma(\mathbf{A})$ being the set of all eigenvalues of $\mathbf{A}$, is smaller than one [4]. In this case $(\mathbf{A}, \mathbf{b})$ serves as fractal code for the signal $\mathbf{x}_f$ which by presumption is the approximation of the original signal $\mathbf{x}$. This is guaranteed by the *collage theorem* [1, 4, 8], which ensures that the fixed point $\mathbf{x}_f$ is close to $\mathbf{x}$ if also the distortion $d(\mathbf{x}, \hat{\mathbf{x}})$ between the original signal and its *collage* $\hat{\mathbf{x}}$ is small. In this case a coding gain is achieved if the transformation can be represented with fewer bits than the signal itself.

Actually the fractal code should be determined in a way that the distortion between the original signal and an appropriated fixed point $d(\mathbf{x}, \mathbf{x}_f)$, which is reconstructed at the decoder, is minimized. Obviously this is a very difficult task from the computational point of view and therefore in practice the above mentioned *collage-coding* has proven well which instead is intended to minimize the distortion between the original signal and the collage $d(\mathbf{x}, \hat{\mathbf{x}})$, though this is not optimal. One possible extension of collage coding yielding a closer fixed-point by recalculating the offset $\mathbf{b}$ for the given linear part $\mathbf{A}$ may be found in [9].

Currently the investigations concerning fractal coding mainly concentrate on library construction and/or signal partitioning in order to minimize the approximation error $d(\mathbf{x}, \hat{\mathbf{x}})$. However the question of convergence is not addressed explicitly in most publications though it is most important for the functionality of these schemes. To overcome this lack, this paper reports on results concerning the convergence problem in fractal coding schemes. This leads to the question whether the eigenvalues of the linear part lie within the unit circle or not. Due to the huge dimension of the transformation matrix $\mathbf{A}$, it is easily understood that straightforward determination of the eigenvalues by solving the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is infeasible. Instead this paper describes how the eigenvalues or at least the spectral radius of the transformation matrix can be determined exactly for a wide range of coding approaches.

The paper is organized as follows: Section 2 introduces the calculation of the eigenvalues of the linear part. In subsection 2.1 it is shown how the eigenvalues of a simple transformation matrix can be derived easily. These results form the basis for a very general class of transformation matrices with which many existing fractal coding schemes can be described exactly when convergence is the issue. The necessary procedures for determining the largest eigenvalue of the transformation matrix is presented in subsection 2.2. The paper concludes with a short summary and a prospect on future investigations.

2. EIGENVALUES OF THE LINEAR PART

Jacquin's original approach [2] is the most general one regarding the variability of the transformation matrix. But due to its complex structure, a simple and general solution for the eigenvalues has not been found yet. Therefore we concentrate our investigations on a slightly simplified version which implies some minor constraints on the single block transformations.

We proceed in two steps: In the first step we investigate a proposal by Monro and Dudbridge for encoding of single image blocks [5]. The structure of the transformation matrix $\mathbf{A}$ becomes rather simple in this case since each block of the signal is treated independently from all others. An exact solution for the eigenvalues of this type of matrix has been presented partly in [10, 11] and therefore topic 2.1 only summarizes the results concisely.

These results serve as basis for the second step on which the main attention is focused. Subsection 2.2 describes a later on published extension [12] including the previous mentioned simple scheme as special case. It additionally allows jointly encoding of signal blocks which makes it very similar to Jacquin's original proposal regarding the structure of the transformation matrix. This transformation is much more complex because of the concatenation of the single block transformations. Coding and decoding of each block now may depend on other parts within the same signal. Hence the single block transformations cannot be performed independently from each other. However this paper presents an algorithm which allows an exact determination of the spectral radius of the transformation matrix during the encoding process and therewith a necessary and sufficient criterion for convergence of the reconstruction process.

2.1 Eigenvalues of the block transformation matrix

According to [5], the encoding process in this simple case starts by segmenting the entire image into $N_R = n/n_R$ non-overlapping *range blocks* of n_R pixels each. The so-called *domain blocks* which also are demanded to be non-overlapping are composed of $m = n_D/n_R,\ m \in \mathbb{N}$ adjacent range blocks. The averaged and subsampled domain block is then mapped onto his underlying range blocks for which according to (1) the scaling- and offset parameters α_{ij} and b_i are determined. It can easily be shown, that the transformation matrix $\mathbf{A}$ for the entire signal $\mathbf{x}$ then is a block-diagonal matrix consisting of N_D square submatrices $\mathbf{A}_i$ and has the following structure:

$$\mathbf{A} = \text{diag}(\mathbf{A}_i)\ ;\ \mathbf{A}_i = \frac{1}{m}\begin{pmatrix} \overbrace{\begin{matrix}\mathbf{a}_{i1} & \mathbf{0} & \cdots & \mathbf{0}\end{matrix}}^{n_R \text{ block columns}} \\ \begin{matrix}\mathbf{0} & \mathbf{a}_{i1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{a}_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_{im} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_{im} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{a}_{im}\end{matrix}\end{pmatrix} \tag{3}$$

with $\mathbf{a}_{ij} = (\alpha_{ij}, \alpha_{ij}, \ldots, \alpha_{ij}),\ \mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^m$ $\forall\, (i \in \{1, 2, \ldots, N_D\}, j \in \{1, 2, \ldots, m\})$. As can be seen, the column sum of $\mathbf{A}_i$ is constant and therefore the spectral radius is bounded by $r_\sigma(\mathbf{A}_i) \leq \frac{1}{m}\sum_{j=1}^{m} |\alpha_{ij}|$ for arbitrary n_D, n_R. Equality holds for non-negative $\mathbf{A}_i$, this is if all $\alpha_{ij} \geq 0$.

For the special case that both n_R and n_D are integral powers of 2, or that m is an integral multiple of n_R which are in practice only minor restrictions, an exact eigenvalue determination may be performed for arbitrary α_{ij}. In this case the characteristic determinant

$$\det(\mathbf{A}_i - \lambda_i\mathbf{I}) = \begin{vmatrix} -\lambda_i & & & & * \\ 0 & -\lambda_i & & & \\ 0 & 0 & \ddots & & \\ \vdots & & \vdots & -\lambda_i & \\ 0 & \cdots & \cdots & 0 & \left(\sum_{j=1}^{m} \frac{\alpha_{ij}}{m}\right) - \lambda_i \end{vmatrix} \tag{4}$$

for calculation of the eigenvalues can be transformed by some elementary operations into an upper triangular form, as outlined in Appendix I. The characteristic equation is the product of the diagonal elements $(-\lambda_i)^{m-1}\left(\left(\sum_{j=1}^{m}\frac{\alpha_{ij}}{m}\right)-\lambda_i\right)\stackrel{!}{=}0$ resulting in the $(m-1)$-fold eigenvalue $\lambda_{i_1,i_2,\ldots,i_{m-1}}=0$ and one single eigenvalue $\lambda_{i_m}=\frac{1}{m}\sum_{j=1}^{m}\alpha_{ij}$. The spectral radius of the entire transformation matrix $\mathbf{A}$ is then exactly determined by

$$r_\sigma(\mathbf{A}) = \max_{i\in\{1,2,\ldots,N_D\}} r_\sigma(\mathbf{A}_i) = \max_i \frac{1}{m}\left|\sum_{j=1}^{m}\alpha_{ij}\right|. \quad (5)$$

It can be seen from (5) that a convergent reconstruction process can be obtained even if some of the scaling parameters α_{ij} exceed the bound of 1.0 which can be derived by examination of the max-norm $\|\mathbf{A}\|_{\max}$ of the transformation matrix [2]. The relationship between any norm, the spectral radius, and Fisher's *eventually contractive maps* [8] is outlined in Appendix II.

2.2 Eigenvalues of a general transformation matrix

In the former described simple case the relationships between the range- and domain blocks are fixed, this is given any arbitrary range block the appropriate domain block is known a priori. Instead the more general transformation addressed in this subsection leaves more freedom for the encoding process since for any given collection of m adjacent range blocks a domain block from a large set of possible ones is selected which fits best in the sense of a distortion measure. As a result of this extension the transformation of the entire signal cannot be decomposed into single block transformations as demanded previously in subsection 2.1, because a block not necessarily is mapped onto itself but may be mapped onto another block within the same signal. This leads to the transformation matrix $\mathbf{A}$ having a more complex structure compared to (3) since the matrix is no longer of block diagonal type. In general it consists of exactly one submatrix $\mathbf{A}_i \neq \mathbf{0}$ in each block row but at any arbitrary block column. Due to this some block columns may only consist of zero matrices $\mathbf{0}$. Those blocks are not part of the iteration process and therefore do not contribute to the largest eigenvalue since the accompanying domain-block is never used for the mapping procedure. Analogous holds for the block rows describing the corresponding range blocks. For determination of the largest eigenvalue these columns and rows can therefore be removed iteratively. The reduced matrix $\mathbf{A}_\mathrm{r}$ obtained in this way describes only those parts of the transformation which influence the convergence. The result of the reduction algorithm is a block matrix with exactly one non-zero submatrix in each block row and block column. By simultaneous exchanging of rows and columns, which does not affect the eigenvalues, the reduced matrix

$$\mathbf{A}_\mathrm{r} = \begin{pmatrix} \mathbf{Z}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}^{(2)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{Z}^{(K)} \end{pmatrix} \quad (6)$$

may be decomposed into a block-diagonal matrix with K non-zero diagonal elements $\mathbf{Z}^{(k)}$. Each of these matrices on the block-diagonal of $\mathbf{A}_\mathrm{r}$ represents one of K parts of the (reduced) signal which can be treated independently from all other parts. The collection of all K parts together with the non-iterative ones then form the entire signal.

For non-overlapping domain blocks as is also presumed in subsection 2.1 the single matrices $\mathbf{Z}^{(k)}, k=1,2,\ldots,K$ may be transformed by simultaneous exchange of rows and columns into the structure

$$\mathbf{Z}^{(k)} = \begin{pmatrix} \mathbf{0} & & & \cdots & \mathbf{A}_1^{(k)} \\ \mathbf{A}_2^{(k)} & \mathbf{0} & & \cdots & \\ \mathbf{0} & \mathbf{A}_3^{(k)} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{A}_{L(k)}^{(k)} & \mathbf{0} \end{pmatrix}. \quad (7)$$

One can see that each matrix $\mathbf{Z}^{(k)}$ describes a closed cycle of block mappings within the k-th part of the signal. A full cycle itself consists of $L(k)$ single block mappings each represented by a matrix $\mathbf{A}_i^{(k)}, i=1,2,\ldots,L(k)$ which has the structure of $\mathbf{A}_i$ as outlined in (3).

The spectral radius of the matrix $\mathbf{Z}^{(k)}$ now determines whether the reconstruction process of the corresponding k-th part of the signal converges or not. Therefore convergence for the entire signal can be achieved if and only if the reconstruction of all K parts of the signal converges. This leads to the problem of calculating the spectral radius for all matrices $\mathbf{Z}^{(k)}$. The suffix k describing the k-th part of the signal is omitted in the following for the purpose of simplification unless otherwise noted.

In order to determine the largest eigenvalue of $\mathbf{Z}$, the l-th power $\mathbf{Z}^l$ is examined. It can be shown easily that for $l=L$ the matrix

$$\mathbf{Z}^L = \mathrm{diag}(\mathbf{D});\ \mathbf{D} = \prod_{l=1}^{L}\mathbf{A}_l \quad (8)$$

is a block diagonal matrix with identical diagonal elements $\mathbf{D}$. Since in all matrices $\mathbf{A}_l$ the column sum $\rho(\mathbf{A}_l)=\sum_{j=1}^{m}\frac{\alpha_{lj}}{m}$ is constant, also the product matrix $\mathbf{D}$ has a constant column sum. For the considered special case that n_R and n_D are integral powers of 2 or m is an integral multiple of n_R, the column sum $\rho(\mathbf{D})=\prod_{l=1}^{L}\rho(\mathbf{A}_l)$ is the product of all column sums $\rho(\mathbf{A}_l)$ and equals the spectral radius of $\mathbf{Z}^L$ so that

$$r_\sigma\left(\mathbf{Z}^L\right) = \left|\prod_{l=1}^{L}\sum_{j=1}^{m}\frac{\alpha_{lj}}{m}\right|. \quad (9)$$

The spectral radius $r_\sigma\left(\mathbf{Z}^L\right)$ of the power matrix $\mathbf{Z}^L$ is equal to the L-th power $\left(r_\sigma(\mathbf{Z})\right)^L$ of the spectral radius of the matrix $\mathbf{Z}$. Consequently the spectral radius $r_\sigma(\mathbf{Z})$ is the L-th root of the column sum $\rho(\mathbf{D})$ of the single block transformations which are involved in the considered cycle, so that

$$r_\sigma(\mathbf{Z}) = \left(r_\sigma\left(\mathbf{Z}^L\right)\right)^{1/L} = \left|\prod_{l=1}^{L}\sum_{j=1}^{m}\frac{\alpha_{lj}}{m}\right|^{1/L}. \quad (10)$$

By reintroducing the suffix k, the condition $r_\sigma\left(\mathbf{Z}^{(k)}\right) < 1$ is necessary and sufficient for a convergent reconstruction of the k-th part of the signal. Otherwise, if $r_\sigma\left(\mathbf{Z}^{(k)}\right) \geq 1$, the reconstruction of this part of the signal is divergent and no unique fixed-point is guaranteed. Therefore convergence in a global sense is ensured if the spectral radius of the reduced matrix $\mathbf{A}_\Gamma$ representing the iterative part of the transformation of the entire image

$$r_\sigma(\mathbf{A}_\Gamma) = \max_{k\in\{1,2,\ldots,K\}} r_\sigma\left(\mathbf{Z}^{(k)}\right) = \max_{k\in\{1,2,\ldots,K\}} \frac{1}{m}\left[\prod_{l=1}^{L(k)}\left|\sum_{j=1}^{m}\alpha_{lj}^{(k)}\right|\right]^{1/L(k)} \tag{11}$$

is smaller than one.

3. SUMMARY AND CONCLUSION

In this paper a new result concerning the convergence properties of fractal coding schemes was presented. A necessary and sufficient condition for convergence is the fact that all eigenvalues of the linear part of the transformation lie within the unit circle. In a first step the eigenvalues for a rather simple class of transformation matrices has been determined. Using this result the second step consisted in a generalization of the structure of the transformation matrix which enables to describe a wide class of fractal coding schemes with respect to their convergence properties. The necessary calculation of the spectral radius for this purpose can be done very easily by evaluating the scaling coefficients.

The presented results also allow to judge existing fractal coding schemes with respect to their convergence. Some investigations concerning this topic are published in [13]. Future work is directed to a more general description of convergence which also intends to include Jacquin's original approach.

4. APPENDIX I

By use of the algorithm outlined in pseudo code below, the matrix $\mathbf{E}_i = \mathbf{A}_i - \lambda_i\mathbf{I} = (e_{jk})_i$ with $\mathbf{A}_i$ from (3) and $\mathbf{I}$ being the identity matrix can be transformed into an upper tridiagonal form for which the determinant can be calculated very easily.

```
For j=1:N_D-1
      column(j) := column(j) - column(j+1);
end

For j=2:N_D
      row(j) := row(j) + row(j–1);
end

For j=2:N_D
      For k=1:j-1
            If (e_jk)<>0) then
                  swap column j with column k
                  swap row j with row k
            end
      end
end
```

5. APPENDIX II

Most investigations concerning the convergence properties of fractal coding schemes are based upon some kind of norms $\|\cdot\|$ of the transformation matrix $\mathbf{A}$ which leads to stronger constraints for the scaling coefficients compared to those derived from the spectral radius as shown above. Consequently it has been observed that convergence was obtained even if those stronger constraints have not been met. Fisher et al. [8] termed this behavior *eventually contractive*, this is if the p-fold application of the transformation is contractive. By regarding the operator norm one can show that an eventually contractive mapping with exponent p is characterized by $\|\mathbf{A}^p\| < 1$. By use of the relationship between any norm and the spectral radius of a matrix

$$r_\sigma(\mathbf{A}) \le \|\mathbf{A}^p\|^{1/p};\ r_\sigma(\mathbf{A}) = \lim_{p\to\infty}\|\mathbf{A}^p\|^{1/p} \tag{12}$$

it can be seen that the case of eventually contractive mappings is included in the criterion based on the spectral radius.

6. REFERENCES

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