European Signal Processing Conference, EUSIPCO'94, Edinburgh, Scotland, 13.-16.09.1994

Modelling of fractal coding schemes

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Abstract - Fractal techniques applied in the area of signal coding suffer from the lack of an analytical description, especially if the question of convergence at the decoder is addressed. For this purpose an expensive eigenvalue calculation of the transformation matrix during the encoding process is necessary which is in general computational infeasible. In this paper the conditional equation for the eigenvalues is determined for a rather general coding scheme. This allows formulation of the probability density function of the largest eigenvalue which in turn determines whether or not the reconstruction converges. These results are not only important for evaluation of the convergence properties of various coding schemes but are also valuable for optimizing their appropriate encoding parameters.

1. Introduction

Fractal techniques are known for several years, especially in three distinct fields of applications namely segmentation, signal modelling and coding. Our attention is focused on the aspects of signal coding for which originally Barnsley [1, 2] contributed major ideas. The lack of a practical algorithm suitable for automatic encoding and modelling of gray-level images at common compression ratios has been filled by Jacquin [3], who proposed a block-based implementation. A recent review on fractal coding schemes may be found in [4] and an excellent mathematical foundation based on the theory of finite-dimensional vector spaces in [5].

Numerous supplements and improvements of Jacquin's scheme have been reported. The objective of this paper is to provide a method for judging the different proposals and to estimate the influence of distinct design parameters on the convergence property of the decoding process. For this purpose a probabilistic approach based upon the transformation matrix is introduced. As derived below, a necessary and sufficient condition for a convergent decoding process is that all eigenvalues of the transformation matrix lie within the unit circle. Since the calculation of the eigenvalues is a great expenditure, it is in general infeasible to perform this during the encoding process. Nevertheless quantification of the probability of divergence, given the type of coding scheme and its pertinent design parameters, is desirable. This way one can quantify the probability of divergent transformations without actually determining the eigenvalues. This is performed by regarding the eigenvalues as random variables and modelling their probability density function (pdf) which depends upon the used coding scheme and its appropriate design parameters.

The paper is organized as follows: The mathematical background of fractal coding is introduced in section 2. Section 3 is concerned with the distribution of the eigenvalues and is divided into three parts. While the topics 3.1 and 3.2 treat two special cases for which a rather simple analytical solution of the characteristic equation is possible, topic 3.3 deals with a more general case, for which no analytical description has been found yet. The paper concludes with a short summary.

2. Theory

The basic coding principle emerges from a blockwise defined affine transformation which composes a signal by parts of itself in order to exploit the natural self-similarities as a special form of redundancy for compression purposes. Fractal coding schemes can be viewed as some sort of vector quantization with a signal dependent codebook.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ be a signal vector consisting of n samples $x_1, x_2, \ldots, x_n$. This signal is segmented into $N_R = n/n_R$ non-overlapping blocks $\mathbf{x}_i$ with n_R elements. Then for each of these blocks one codebook entry $\mathbf{y}_j$ from a set of N_D entries is selected which after scaling with α_{ij} and adding an offset $b_i\mathbf{1}$ minimizes some predefined distortion measure

$$\varrho(\mathbf{x}_i, \hat{\mathbf{x}}_i) = \varrho(\mathbf{x}_i, \alpha_{ij}\mathbf{y}_j + b_i\mathbf{1}) \tag{1}$$

for all blocks $\mathbf{x}_i$. The codebook is generated from the signal $\mathbf{x}$ itself by use of a codebook construction matrix $\mathbf{C}$ which is mainly determined by the type of coding scheme. If $\mathbf{F}_j$ denotes the 'fetch-operation' of the codebook entry $\mathbf{y}_j = \mathbf{F}_j\mathbf{C}\mathbf{x}$ from the codebook and $\mathbf{P}_i$ the 'put-operation' of the modified codebook entry $\hat{\mathbf{x}}_i = \alpha_{ij}\mathbf{y}_j + b_i\mathbf{1}$ into the approximation $\hat{\mathbf{x}}$, the mapping process for the entire image $W: \mathbb{R}^n \to \mathbb{R}^n$ may be formulated by

$$\begin{aligned}\hat{\mathbf{x}} &= \sum_{i=1}^{N_R} \mathbf{P}_i(\alpha_{ij}\mathbf{F}_j\mathbf{C}\mathbf{x} + b_i\mathbf{1}) \\ &= \left\{\sum_{i=1}^{N_R} \mathbf{P}_i\alpha_{ij}\mathbf{F}_j\mathbf{C}\right\}\mathbf{x} + \sum_{i=1}^{N_R} \mathbf{P}_i b_i\mathbf{1} \\ &= \mathbf{A}\mathbf{x} + \mathbf{b} = W(\mathbf{x})\end{aligned} \tag{2}$$

This represents an affine transformation consisting of a linear part $\mathbf{A}$ and a non-linear offset $\mathbf{b}$ which together form the *fractal code* $(\mathbf{A}, \mathbf{b})$ for the original signal. A very simple coding scheme may be constructed if the mapping is *contractive* or at least *eventually contractive*. This is if there exists a number $s_q < 1, s_q \in \mathbb{R}$ so that for some $0 < q \leq Q < \infty, q \in \mathbb{N}$ the contractivity condition

$$\varrho(W^q(\mathbf{x}), W^q(\mathbf{y})) \leq s_q \varrho(\mathbf{x}, \mathbf{y}) \, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{3}$$

holds. Then the decoder can construct a unique *fixed point* $\mathbf{x}_f$ from the fractal code without any knowledge about the codebook. This is guaranteed by *Banach's fixed point theorem* which states that the sequence of iterates $\{\mathbf{x}_k\}$ with $\mathbf{x}_{k+1} = W(\mathbf{x}_k)$ converges for any arbitrary initial signal $\mathbf{x}_0 \in \mathbb{R}^n$ to a unique fixed point $\mathbf{x}_f \in \mathbb{R}^n$ of W. If contractivity is assumed, the *collage theorem* (e.g. [5, 6]) ensures that the fixed point itself is close to the original signal, since

$$\varrho(\mathbf{x}, \mathbf{x}_f) \leq \frac{1 - s_1^Q}{(1 - s_1)(1 - s_Q)} \varrho(\mathbf{x}, \hat{\mathbf{x}}). \tag{4}$$

As can be seen from (4) the collage theorem also motivates the mapping process at the encoder which minimizes the approximation error $\varrho(\mathbf{x}, \hat{\mathbf{x}})$. Because not the original signal $\mathbf{x}$ itself, but a fixed point $\mathbf{x}_f$, which is close to the original signal, is encoded, fractal coding schemes sometimes are also termed *attractor coding*. A coding gain in this way can be achieved, if the fractal code $(\mathbf{A}, \mathbf{b})$, which serves as representation of the fixed point in the fractal domain, can be expressed with fewer bits than the original signal itself.

For affine transformations can be shown that a necessary and sufficient condition for contractivity is that the spectral radius $r_\sigma(\mathbf{A})$ of the linear part, which is the largest eigenvalue in magnitude, is smaller than one [5, 7]. One can show that this demand is equivalent with the statement of eventual contractivity.

Control about the convergence of W can therefore be obtained by determining the eigenvalues of the linear part $\mathbf{A}$ which is in general a very difficult task and analytical solutions are given only for some rather simple coding schemes, e.g. [7, 8, 5]. Contractivity is always ensured, if the magnitude of all scaling coefficients α_{ij} is strictly smaller than one. As reported by several authors, e.g. [9, 10], a less stringent restriction for the α_{ij} improves reconstruction quality and convergence speed. On the other hand contractivity of the transformation W is no longer guaranteed. Our investigations have shown, that the scaling coefficients may be regarded as statistically independent and in $[-\alpha_{\max}; \alpha_{\max}]$ uniformly distributed random variables. Since the eigenvalues are solely determined by the scaling coefficients and the structure of the linear part they are also random variables. The following section illustrates by means of three distinct applications how the probability density function (pdf) of the eigenvalues for various choices of the design parameters can be derived. Whereas for the first two schemes, described in topic 3.1 and 3.2, an analytical solution of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) \stackrel{!}{=} 0$ is given, for the later and more general scheme the pdf is approximated. The pdf of the eigenvalues determines the probability for divergent transformations and so the influence of various design parameters on the contractivity can be quantified.

3. Applications

As mentioned above, contractivity is determined by the largest eigenvalue in magnitude of the transformation matrix $\mathbf{A}$. Due to its huge dimension straightforward determination by solving the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is infeasible. Instead the specific structure of the matrix must be considered in order to find an exact and quick solution. For a rather general coding approach this is done in [8], so this paper only concisely summarizes the results.

We emerge from a coding approach published in [11]. The basic idea is to find for m consecutive blocks within the signal $\mathbf{x}_i, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{i+m-1}$ each consisting of n_R samples another block $\mathbf{y}_j$ of size mn_R samples, so that after some sort of geometric transformation, scaling and an additional offset the distortion measure (1) after the mapping process $\varrho(\mathbf{x}_i, \alpha_{ij}\mathbf{y}_j + b_i\mathbf{1})$ becomes as small as possible. As shown in [8] the largest eigenvalue of the transformation matrix for the entire signal is then determined by

$$|\lambda(\mathbf{A})|_{\max} = \max_{k \in \{1,2,\ldots,K\}} \frac{1}{m} \left[\prod_{l=1}^{L(k)} \left| \sum_{j=1}^{m} \alpha_{lj}^{(k)} \right| \right]^{1/L(k)}. \tag{5}$$

The index k denotes one of K mapping cycles. Each of these cycles can be treated independently from all others. This is due to the fact, that those part of the signal belonging to one cycle is not regarded as codebook entry from all other cycles and vice versa. The number $L(k)$ is the length of cycle k, which equals the number of codebook entries involved in this cycle.

For simplification this paper considers two important special cases from the literature. The first one, published in [12], treats each m blocks of the signal independently from all others. Therefore the length of mapping cycles $L(k)$ for all mappings equals one. We call these schemes *non-concatenated coding schemes*. The second one does not geometrically scale the codebook entries, or only does this by subsampling. One can show that this results in the parameter m being equal to one. Those schemes have been thoroughly investigated in [5] and are called *decimating coding schemes*.

3.1 Non-concatenated coding scheme

The coding schemes described in this section are characterized by cycles with length one. Then eq. (5) can be simplified, so that the eigenvalue for one cycle is determined by

$$|\lambda(\mathbf{A})|_{\text{cycle=k}} = \frac{1}{m} \left| \sum_{j=1}^{m} \alpha_{1j}^{(k)} \right|. \tag{6}$$

As presumed above, the scaling coefficients are uniformly distributed in $[-\alpha_{\max}; \alpha_{\max}]$. If they are also statistically independent, the pdf $p_\lambda(\xi)$ of the eigenvalues is equal to the m-fold convolution of the pdf $p_\alpha(\xi)$ of the scaling parameter. Introducing the rect-function with

$$\text{rect}(\xi) = \begin{cases} 0 & |\xi| > 1/2 \\ 1/2 & \text{for } |\xi| = 1/2 \\ 1 & |\xi| < 1/2 \end{cases} \tag{7}$$

the pdf of the random variable $y = \alpha/m$ can be written as

$$p_y(\xi) = \frac{m}{2\alpha_{\max}} \text{rect}\left(\frac{\xi m}{2\alpha_{\max}} \right). \tag{8}$$

The m-fold convolution of $p_y(\xi)$ with itself then yields the pdf of the eigenvalues, which is

$$p_\lambda(\xi) = \underbrace{p_y(\xi) * p_y(\xi) * \cdots * p_y(\xi)}_{\text{m-times}} . \tag{9}$$

The mapping $W : \hat{\mathbf{x}} \rightarrow \mathbf{A}\mathbf{x} + \mathbf{b}$ results in a divergent reconstruction, if and only if the largest eigenvalue of the linear part $\mathbf{A}$ is outside the unit circle. Since p_λ is an even function, the probability for divergence can easily be determined by

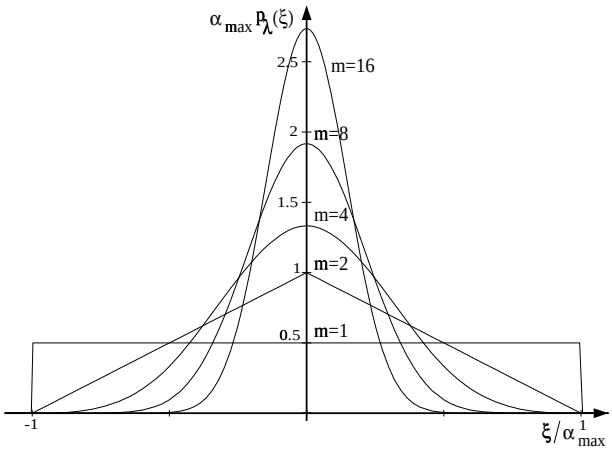

Figure 1 Probability density function $p_\lambda(\xi)$ for some common values of the design parameter m

$$\text{Prob}\{|\lambda| \geq 1\} = 2\int_1^\infty p_\lambda(\xi)d\xi \tag{10}$$

Figure 2 shows this probability for some different parameters m as function of the largest allowed scaling coefficient $\alpha_{\max}$. One can see, that the choice $m = 1$ is disadvantageous in comparison to larger values for m as far as the convergence property is concerned.

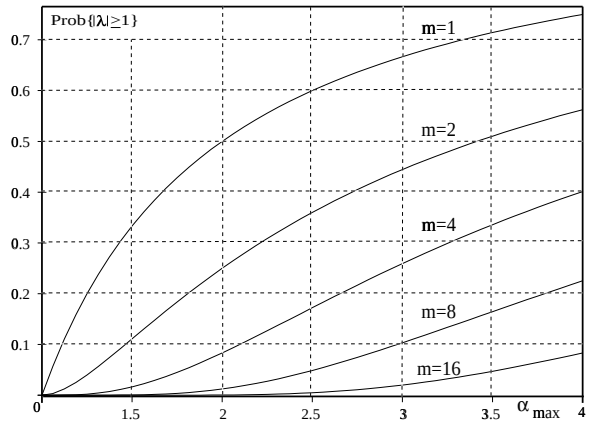

Figure 2 Probability for divergent reconstruction of non-concatenated coding scheme

3.2 Decimating coding scheme

Since the design parameter m equals one, as has been mentioned above, the eigenvalue equation for this type of coding scheme can be simplified to

$$|\lambda(\mathbf{A})|_{\text{cycle=k}} = \left(\prod_{l=1}^{L(k)} \left| \alpha_{l1}^{(k)} \right| \right)^{1/L(k)} . \tag{11}$$

In order to determine the divergence probability Prob$\{|\lambda| \geq 1\}$ the pdf $p_{|\lambda|}(\xi)$ has to be calculated. For simplification a new random variable $\mu = \left|\lambda^L\right|, L \in \mathbb{N}$ is introduced. The scaling coefficients are assumed to be statistically independent and uniformly distributed in the interval $[-\alpha_{\max}; \alpha_{\max}]$, so that the pdf of the product is determined by

$$p_\mu(\xi) = \text{rect}\left(\frac{\xi}{\alpha_{\max}^L} - \frac{1}{2} \right) \frac{\left(\ln\left(\frac{\alpha_{\max}^L}{\xi} \right) \right)^{L-1}}{(L-1)!\alpha_{\max}^L} . \tag{12}$$

Since Prob$\{|\lambda| \geq 1\}$ = Prob$\{\mu \geq 1\}$ the probability for divergence can be obtained by integrating the pdf $p_\mu(\xi)$ so that finally

$$\text{Prob}\{|\lambda| \geq 1\} = 1 - \frac{1}{\alpha_{\max}^L} \sum_{l=0}^{L-1} \frac{L^l}{l!} (\ln \alpha_{\max})^l . \tag{13}$$

As can be seen from figure 3 long mapping cycles are advantageous for a convergent reconstruction process.

Summarizing the results of topic 3.1 and 3.2, one can state that a fractal coding scheme which is optimized with respect to the contractivity of the transformation should combine a geometrical scaling of the codebook entries ($m > 1$) with long mapping cycles ($L(k) > 1$). A typical representative of such a scheme is described in the following topic.

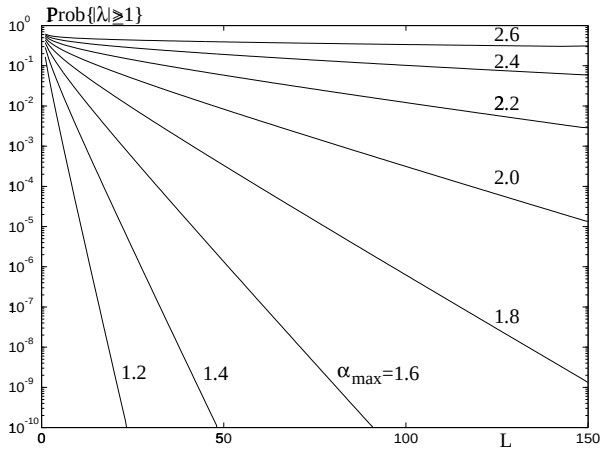

Figure 3 Probability for divergent reconstruction of decimating coding scheme

3.3 General coding scheme

Jacquin's original proposal for encoding of gray-level images is the most general one as far as the possible variations of the transformation matrix $\mathbf{A}$ are concerned. To the knowledge of the authors no exact formulation of the eigenvalues has been found yet, but only an upper bound derived from the euclidean operator norm [13]. So an analytical derivation of the pdf of the eigenvalues, as has been performed in the previous two topics, is not possible. Instead the pdf of the largest eigenvalue is approximated. This has been done by generating the transformation matrix $\mathbf{A}$ and evaluating its largest eigenvalue. For a large number of experiments the probability of divergence approximates

$$\text{Prob}\{|\lambda| \geq 1\} \approx \frac{n_{|\lambda|\geq 1}}{n},\ n \text{ large} \qquad (14)$$

with $n_{|\lambda|\geq 1}$ denoting the number of experiments where the magnitude of the largest eigenvalue exceeds one. Results of our computer simulations are depicted in figure 4. One can see that a simple decimation matrix $m = 1$ results in significantly more divergent transformations compared to a matrix which performs averaging of two or more samples $m > 1$. The differences between larger m are less significant, the choice of a suited averaging parameter is therefore mainly determined by the associated computational burden of the encoding process.

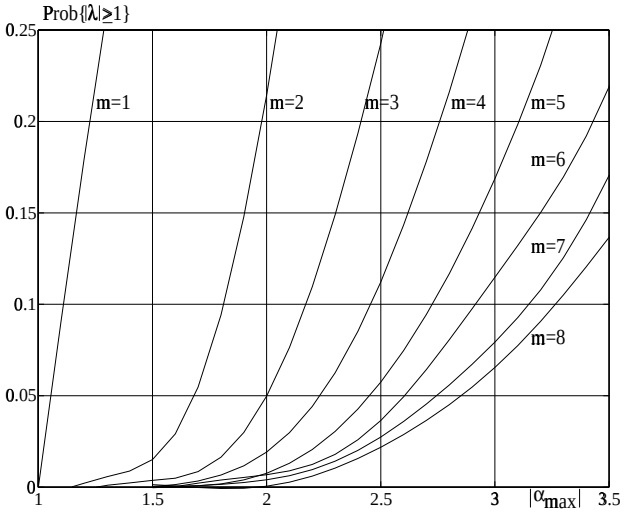

Figure 4 Probability for divergent reconstruction of Jacquin's coding scheme

4. Summary

In this paper the transformation matrices of fractal coding schemes are examined. Since all eigenvalues of the transformation matrix must lie within the unit circle, the probability for divergent reconstruction sequences at the decoder can be quantified by determining the pdf of its eigenvalues.

The conditional equation for the eigenvalues has been given for a rather general class of coding schemes. By modelling the scaling parameters as uniformly distributed, statistically independent random variables the eigenvalues itself are functions of random variables. Their pdf's have been modelled for the two special cases of non-concatenated and decimating coding schemes. This allows specification of the probability for divergent reconstruction at the decoder and an appropriate choice for some design parameters. Up to now no simple conditional equation for the eigenvalues of Jacquin's original proposal has been found. Therefore their appropriate pdf is approximated by the relative frequency of eigenvalues being larger than one as result of an experiment performed many times.

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