ON THE CONVERGENCE OF FRACTAL TRANSFORMS

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ABSTRACT

This paper reports on investigations concerning the convergence of *fractal transforms* for signal modelling. Convergence is essential for the functionality of fractal based coding schemes. The coding process is described as non-linear transformation in the finite-dimensional vector space. Using spectral theory, a necessary and sufficient condition for the contractivity is derived from the eigenvalues of a special linear operator. In the same way some constraints for the choice of the encoding parameters are deduced which are less strict than those imposed so far. The proposed contractivity measure can be calculated directly from the transformation parameters during the encoding process. For complex encoding schemes the calculation of the eigenvalues may be infeasible. For those cases a contractivity criterion derived from the norm of the operator is suggested.

1. INTRODUCTION

The main purpose of fractal coding schemes is to exploit self-similar structures in natural signals for data compression purposes. Self-similarity in the way we will define it may be viewed as a special kind of redundancy. In contrast to common transformations e.g. DCT, whose coding gain is due to the correlations of adjacent samples of the signal, fractal coding schemes mainly exploit the *long range correlations* within the signal.

The basic idea for encoding of images using fractal techniques has been reported by Barnsley [1, 2, 3]. A first practical implementation capable of encoding grey-scale images at common compression ratios has been proposed by Jacquin [4, 5, 6]. Some improvements and modifications, e.g. [7, 8, 9, 10, 11] have been published, but the underlying concept of blockwise approximation of the image by parts of itself is still the same. A review on fractal image coding may be found in [12].

A signal $\mathbf{x}$ is denoted *self-similar* if all its parts approximately resemble other parts of the same signal using only scaling, rotating, mirroring, and shifting operations. A signal of this kind then fulfills the invariance condition

$$\mathbf{x} \simeq W(\mathbf{x}) \tag{1}$$

with W representing the allowed operations. A very simple method for reconstructing the signal $\mathbf{x}$ from the transformation W can be stated if W is *contractive*. Then $\mathbf{x}$ can be viewed as the fixed point of the transformation W and according to *Banach's fixed point theorem* it may be reconstructed solely from the knowledge of the transformation W. A coding gain is achieved, if the transformation can be represented with fewer bits than the signal itself.

To the knowledge of the authors, Lundheim was the first who analyzed fractal signal modelling for discrete signals in the finite dimensional vector space by use of functional analysis. An excellent introduction in this field containing some new results may be found in [13].

By describing the coding scheme as affine transformation in the finite-dimensional vector space $\mathbb{R}^n$, we obtain some conditions for the choice of the encoding parameters which are tighter than the ones imposed so far. Utilizing these results, existing approaches can be improved in terms of convergence speed, reconstruction quality, and compression ratio.

After reviewing the theory of normed metric spaces, a necessary and sufficient criterion for the contractivity of the used transformation is derived from the eigenvalues of the corresponding operator which also implies constraints for the encoding parameters. As applications, two fractal coding implementations for grey-scale images are discussed. The direct calculation of the eigenvalues is feasible for rather simple coding schemes only, whereas for a more complex scheme a norm based criterion is proposed. The summary concludes with some experimental results obtained for natural grey-scale images.

2. THEORY

Consider a signal $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ consisting of n samples as point in the n-dimensional vector space $\mathbb{R}^n$. The components $x_i;\ 1 \leq i \leq n;\ x_i \in \mathbb{R}$ represent the sample values. In order to measure the distance of a point $\mathbf{x}$ within this space from the zero-point $\mathbf{0} = (0, 0, \cdots, 0)^T$ several *norms*, denoted by $\|\mathbf{x}\|$, are defined. For our applications the *Euclidean norm*

$$\|\mathbf{x}\| := \sqrt[+]{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2} \tag{2}$$

defined by the square root of the inner product $\langle \mathbf{x}, \mathbf{x} \rangle$ is the most suited one. By this definition $\mathbb{R}^n$ becomes a normed space and by inducing a metric

$$\varrho(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|,\ \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{3}$$

a normed metric space denoted by $(\mathbb{R}^n, \varrho)$. Transformations within this space can be described by linear operators. Therefore a suited *operator-* or *matrix norm* is required. One which is consistent with the Euclidean vector norm in the sense that $\|\mathbf{A}\mathbf{x}\| \le \|\mathbf{A}\|\,\|\mathbf{x}\|$ holds, is the so called *spectral-* or *Hilbert norm*, defined by

$$\|\mathbf{A}\|_{sp} := \sup_{\lambda \in \sigma(\mathbf{A}^T\mathbf{A})} \sqrt{|\lambda|}\,. \tag{4}$$

It is the square root of the largest eigenvalue in magnitude of the matrix $\mathbf{A}^T\mathbf{A}$. Additionally for every linear operator $\mathbf{A}$ the *spectral radius* $r_\sigma(\mathbf{A})$ is defined by

$$r_\sigma(\mathbf{A}) := \sup_{\lambda \in \sigma(\mathbf{A})} |\lambda| \tag{5}$$

which is the magnitude of the largest eigenvalue of $\mathbf{A}$. Any norm $\|\mathbf{A}\|$ and spectral radius r_σ of a linear operator $\mathbf{A}$ are connected through the following equations:

$$\begin{aligned} r_\sigma(\mathbf{A}) &\le \|\mathbf{A}\| \\ r_\sigma(\mathbf{A}) &= \lim_{j\to\infty} \sqrt[j]{\|\mathbf{A}^j\|}\,. \end{aligned} \tag{6}$$

2.1 Fractal encoding

All existing practical implementations for block-oriented fractal coding schemes emerge from an affine transformation which is capable of performing the scaling, rotating, mirroring, and shifting operations in order to exploit the presupposed self-similarities of the signal. An affine transformation W of the entire signal $\mathbf{x}$ is defined by

$$W:\ \mathbb{R}^n \to \mathbb{R}^n \ \Rightarrow\ \mathbf{x} \to \mathbf{A}\mathbf{x}+\mathbf{b} \tag{7}$$

consisting of a linear part $\mathbf{A}\mathbf{x}$ and an additive part $\mathbf{b}$. The transformation W is called *contractive* if there exists a constant $s < 1$, such that

$$\varrho\,(W(\mathbf{x}), W(\mathbf{y})) \le s\ \varrho(\mathbf{x},\mathbf{y})\ \forall\, \mathbf{x},\mathbf{y} \in \mathbb{R}^n. \tag{8}$$

With the definition of the metric (3), the affine transformation (7), and the contractivity (8) we obtain the sufficient condition

$$\|\mathbf{A}\| \le s < 1 \tag{9}$$

for contractivity in the sense of any norm.

The encoding process for a given signal $\mathbf{x}$ now consists in finding a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ fulfilling the invariance condition (1) at least approximately. This means, that the approximation error

$$\varrho(W(\mathbf{x}),\mathbf{x}) = \varrho(\mathbf{A}\mathbf{x}+\mathbf{b},\mathbf{x}) \tag{10}$$

must be small. Additionally it must be assured that W is a contractive transformation in order to get the scheme work, this is if $\|\mathbf{A}\| < 1$ holds. Data compression can then be achieved, if the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ can be stored more efficiently than the signal $\mathbf{x}$ itself.

2.2 Fractal decoding

Banach's fixed point theorem gives us an idea how the decoding process works:

Let $W : \mathbb{R}^n \to \mathbb{R}^n$ be a contractive transformation and $(\mathbb{R}^n, \varrho)$ a metric space with metric ϱ, then the sequence of signals $\{\mathbf{x}_k\}$ constructed by $\mathbf{x}_{k+1} = W(\mathbf{x}_k)$ converges for any arbitrary initial signal $\mathbf{x}_0 \in \mathbb{R}^n$ to the unique fixed point $\mathbf{x}_f \in \mathbb{R}^n$ of the transformation W, i.e.

$$\mathbf{x}_f = W(\mathbf{x}_f) = \mathbf{A}\mathbf{x}_f + \mathbf{b}\,. \tag{11}$$

The reconstruction error $\varrho(\mathbf{x}_f, \mathbf{x})$ between the original signal $\mathbf{x}$ and its fractal reconstruction $\mathbf{x}_f$ is then bounded by

$$\varrho(\mathbf{x}_f,\mathbf{x}) \le \frac{1}{1-\|\mathbf{A}\|}\,\varrho(W(\mathbf{x}),\mathbf{x})\,. \tag{12}$$

The contractivity condition (9) is only sufficient but not necessary for the convergence of the iteration process. A sufficient and necessary condition can be formulated by using the spectral radius of the transformation matrix.

Emerging from any arbitrary initial signal $\mathbf{x}_0$ the decoder iteratively applies the transformation (7). For the k-th iterate then follows:

$$\mathbf{x}_k = W^{\circ k}(\mathbf{x}_0) = \mathbf{A}^k\mathbf{x}_0 + \left(\sum_{i=0}^{k-1} \mathbf{A}^i\right)\mathbf{b}\,. \tag{13}$$

If and only if the spectral radius satisfies $r_\sigma(\mathbf{A}) < 1$, then $\lim\limits_{k\to\infty} \mathbf{A}^k = \mathbf{0}$ and $\lim\limits_{k\to\infty} \sum_{i=0}^{k-1} \mathbf{A}^i = (\mathbf{I}-\mathbf{A})^{-1}$ with the identity $\mathbf{I}$ and the null matrix $\mathbf{0}$ [14]. Thereby the sequence $\{\mathbf{x}_k\}$ converges to the fixed point

$$\mathbf{x}_f = \lim_{k\to\infty} \mathbf{x}_k = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} \tag{14}$$

of the transformation.

According to eq. (6) the spectral radius $r_\sigma(\mathbf{A})$ is a lower bound for every norm $\|\mathbf{A}\|$. In literature the special case $r_\sigma(\mathbf{A}) < 1 \le \|\mathbf{A}\|$ is termed *eventual contractivity* (see [15, 13] and section 4). This is due to the fact that $\|\mathbf{A}\| < 1$ is only a sufficient but not necessary condition for the convergence of the iteration process (13).

3. APPLICATIONS

The most important question concerning fractal coding schemes is whether the reconstruction process is convergent or not. Considering the above definitions of the norm and the spectral radius we investigated this property for two widely known schemes used for encoding of grey-scale images. The first encodes single blocks of the signal individually while the second one encodes a blocks dependent on other parts of the signal. Hence, the first is a special case of the second scheme.

3.1 Encoding a single block

An interesting proposal for encoding of single blocks of a signal has been published by Monro and Dudbridge [8]. The basic idea of the encoding process consists in mapping the entire block consisting of D samples onto $m = D/R$ non-overlapping underlying subblocks each consisting of R samples. The mapping is chosen to minimize the distance between the original block and the approximation according to eq. (10). By using the above notation it can be described by a linear operator

$$\mathbf{A} = \frac{1}{m}\left(\underbrace{\begin{array}{cccc} \overbrace{\alpha_1\cdots\alpha_1}^{\text{m times}} & 0 & \cdots & 0 \\ 0 & \alpha_1\cdots\alpha_1 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_1\cdots\alpha_1 \\ \alpha_2\cdots\alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2\cdots\alpha_2 & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \alpha_2\cdots\alpha_2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_m\cdots\alpha_m & 0 & \cdots & 0 \\ 0 & \alpha_m\cdots\alpha_m & \cdots & \vdots \\ \vdots & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \alpha_m\cdots\alpha_m \end{array}}_{\text{R groups}}\right) . \tag{15}$$

By (4) the spectral norm $\|\mathbf{A}\|_{sp}$ is defined by the largest eigenvalue of the symmetric block diagonal matrix $\mathbf{A}^T\mathbf{A}$. For the linear operator (15) one obtains:

$$\|\mathbf{A}\|_{sp} = \sup_{\lambda\in\sigma\left(\mathbf{A}^T\mathbf{A}\right)} \sqrt{|\lambda|} = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\alpha_i^2}\ . \tag{16}$$

Obviously $\|\mathbf{A}\| < 1$ holds for each $|\alpha_i| < 1 \ \forall\ 1 \le i \le m$, as is presupposed in [8], but the contractivity constraint (16) allows more freedom in the choice of the parameters since it only constrains the square sum of the elements.

Due to the very simple structure of the operator $\mathbf{A}$ in this case, not only the eigenvalues of the block diagonal matrix $\mathbf{A}^T\mathbf{A}$, but also the eigenvalues of $\mathbf{A}$ itself may be calculated. This gives us the possibility to determine the spectral radius $r_\sigma(\mathbf{A})$ in order to obtain a sufficient *and* necessary condition for the convergence of the decoding process. For this purpose the magnitude of the largest eigenvalue of the matrix $\mathbf{A}$ has to be determined. For arbitrary elements $\alpha_i \in \mathbb{R}$ the largest eigenvalue of $\mathbf{A}$ is bounded by the constant sum of the absolute values of any column of $\mathbf{A}$, hence

$$r_\sigma(\mathbf{A}) = \sup_{\lambda\in\sigma(\mathbf{A})} |\lambda| \le \frac{1}{m}\sum_{i=1}^{m}|\alpha_i|\ . \tag{17}$$

Equality is only met if $\mathbf{A}$ is non-negative, this is if $\alpha_i \ge 0 \ \forall\ 1 \le i \le m$. In general, the transformation matrix is not necessarily non-negative. Due to eq. (17), negative elements α_i even improve the convergence. This has also been confirmed experimentally.

3.2 Joined encoding of all blocks

The original proposal from Jacquin [6] is a typical representative of this scheme. The entire signal is partitioned into N_R non-overlapping *range blocks* consisting of R samples. For each of these blocks one of N_D *domain blocks* with size D samples is chosen which fits best in the sense of the distance measure (3). So one sample in the approximation of the range block is determined by the average of $m = D/R$ samples in the domain block. $\mathbf{A}$ now describes the transformation of the whole signal. Therefore it consists of a large number of matrix elements. Hence the direct calculation of the eigenvalues which leads to the spectral radius is computationally prohibitive. Instead the determination of the spectral norm $\|\mathbf{A}\|_{sp}$ is presented for this scheme. For the case of decimating matrices, this is if each row of $\mathbf{A}$ contains one and only one non-zero element (no averaging), Lundheim [13] presented an exact solution for the eigenvalues.

In the special case of Jacquin's proposal one can show without going too much into detail, that the matrix

$$\mathbf{B} = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} \sum\limits_{i\in I(j=1)} \mathbf{A}_{i1}^T\mathbf{A}_{i1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sum\limits_{i\in I\left(j=N_D\right)} \mathbf{A}_{iN_D}^T\mathbf{A}_{iN_D} \end{pmatrix} \tag{18}$$

is a block diagonal matrix with submatrices of size $D \times D$ elements. The matrix

$$\mathbf{A}_{ij} = \frac{1}{m}\left(\underbrace{\begin{array}{cccc} \overbrace{\alpha_i\cdots\alpha_i}^{\text{m times}} & 0 & \cdots & 0 \\ 0 & \alpha_i\cdots\alpha_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_i\cdots\alpha_i \end{array}}_{\text{R groups}}\right) \tag{19}$$

describes the mapping from the j-th domain block onto the i-th range block (see eq. (15)). $I(j)$ is the set of indices of those range blocks onto which the j-th domain block is mapped. The matrix $\sum_{i\in I(j)} \mathbf{A}_{ij}^T\mathbf{A}_{ij}$ has $R-$times the non-zero eigenvalue $\frac{1}{m}\sum_{i\in I(j)}\alpha_i^2$ and $(D-R)-$times the eigenvalue zero. Due to the special structure of the matrix $\mathbf{B}$, the eigenvalues of the submatrices are also the eigenvalues of $\mathbf{B}$ itself. Thereby the matrix $\mathbf{B}$ has the non-zero eigenvalues $\frac{1}{m}\sum_{i\in I(j)}\alpha_i^2 \ \forall\ 1 \le j \le N_D$. This way, the spectral norm of the transformation matrix $\|\mathbf{A}\|_{sp}$ defined by eq. (4) may be calculated by

$$\|\mathbf{A}\|_{sp} = \sup_{1\le j\le N_D} \sqrt{\frac{1}{m}\sum_{i\in I(j)}\alpha_i^2} \tag{20}$$

4. RESULTS AND CONCLUSION

Fisher [10] showed by experimental analysis of Jacquin's coding scheme, that the constraint $|\alpha_i| < 1 \; \forall \; i$ is not necessary in order to obtain a convergent sequence of iterates. He called this behavior the *eventual contractivity* of the transformation $W: \mathbb{R}^n \rightarrow \mathbb{R}^n$. This is if there exists a positive integer p called the *exponent of eventual contractivity* so that the transformation $W^{\circ p}(\mathbf{x}) = \underbrace{W(W(\cdots(\mathbf{x})))}_{\text{p times}}$ is contractive. According to the relationship between the spectral radius and the norm (6) eventual contractivity with exponent p means that $\sqrt[j]{\|\mathbf{A}^j\|} < 1 \; \forall \; j \geq p$. Thereby all matrices $\mathbf{A}^j, \forall \; j \geq p$ are also contractive in the sense of the norm $\|\mathbf{A}^j\|$.

Lundheim [13] proved, that eventual contractivity is not a question of the choice of the norm. This means that if an operator is eventual contractive in one norm, it also is eventual contractive in any other norm. This is due to the fact that from all norms only a sufficient but not necessary conditions for contractivity of the operator can be derived.

Nevertheless not all norms are equally suited for the considered application of signal encoding. Table 1 shows a comparison between the *maximum norm* $\|\mathbf{A}\|_{\max}$ (see e.g. [10]) which is mostly applied due to its ease of computation and the proposed spectral norm $\|\mathbf{A}\|_{sp}$. As can be seen, the usage of the spectral norm results in contractive transformations even if the scaling parameters α_i are not bounded by one. In these cases, which are the most important ones, the maximum norm is useless since it yields divergent operators (in the sense of the norm). A tighter approximation is obtained by the spectral norm. The calculation of the spectral radius offers the best results but this is computationally feasible only for simple encoding schemes as described in topic 3.1.

Test image	$\alpha_{\max}$	$\|\mathbf{A}\|_{max}$	$\|\mathbf{A}\|_{sp}$	$r_\sigma(\mathbf{A})$
LENA	1.0	0.9953	0.7665	0.4989
	1.5	1.4956	0.9120	0.5661
	5.0	4.7582	2.8361	0.6695
BOATS	1.0	0.9995	0.7782	0.5273
	1.5	1.4973	0.9900	0.5687
	5.0	4.9981	2.5559	0.6983

Table 1: Comparison of maximum norm $\|\mathbf{A}\|_{\max}$, the spectral norm $\|\mathbf{A}\|_{sp}$ and the spectral radius $r_\sigma(\mathbf{A})$ for some test images and α-bounds $\alpha_{\max}$.

The consideration of the spectral norm or spectral radius instead of the maximum norm for determining convergence of fractal encoding schemes leaves more freedom in the choice of the encoding parameters, so that reconstruction quality as well as convergence speed can be increased.

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