ATTRACTOR IMAGE CODING USING LAPPED PARTITIONED ITERATED FUNCTION SYSTEMS

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ABSTRACT

The usual attractor coding technique is to partition a given image into a number of non-overlapping range blocks. Each block of the partition is expressed as the contractive transformation of another part of the image. However, the non-overlapping partitioning induces blocking artifacts which is highly disturbing to human visual system. In this work, a novel coding scheme using iterated function systems with lapped range blocks (LPIFS) is proposed. Each range block laps with its adjacent blocks through a weighing window which has diminishing magnitudes towards its borders. In order to avoid blurring of image details, the transformation parameters are computed such that the aliasing error is compensated. The contractivity of the proposed transformation is also proved. Experiments show that a very significant improvement of the visual quality of the decoded images with nearly no loss in image details.

1. INTRODUCTION

Attractor image coding (commonly known as fractal coding) has been successfully applied to encode digital images at low bit rates. The usual coding scheme is to construct the partitioned iterated function systems (PIFS) code for an image [1]. The coding algorithm partitions an image into a number of square blocks (domain blocks). After this, another partition into smaller blocks (range blocks) takes place. For every range block the best matching domain blocks is searched among all domain blocks by performing a set of transformations on the blocks. The compression is obtained by storing only the descriptions of these transformations.

Owing to the partitioning of an image into disjoint range blocks, blocking artifacts which is highly visibly annoying to human visual system occur. One solution to this problem is to allow range blocks to overlap. Reusens proposed to use overlapping blocks and average the overlapping regions during decoding [2]. However, the image details within the overlapping regions are blurred and so the overlapping regions are restricted to the block boundaries only. On the other hand, such blurring can be avoided if the global collage distance for all blocks is minimized simultaneously instead of minimizing collage distance for each block independently [3]. However, a very large system of equations is involved in this optimization.

In this work, we propose a new coding scheme using iterated function systems with lapped range blocks (LPIFS). Each range block laps with its adjacent blocks through a weighing window which is diminishing in magnitudes towards its boundaries. The transformation parameters are computed in such a way that the aliasing error is compensated. The image details are preserved and no large system of equations is needed.

2. THEORETICAL FOUNDATION

Let (M, d) be a metric space whose elements are digital images. The distance between any two images, $\mathcal{I}_1 = \{i^{(1)}_{u,v}\}$ and $\mathcal{I}_2 = \{i^{(2)}_{u,v}\}$, in (M, d) is given by

$$d(\mathcal{I}_1, \mathcal{I}_2) = \sqrt{\sum_{u,v} (i^{(1)}_{u,v} - i^{(2)}_{u,v})^2} \tag{1}$$

where u, v denote the vertical and horizontal pixel coordinates respectively. A transformation $\mathbf{T} : M \to M$ is said to be a contraction if $\exists s \in (0,1)$ s.t. $\forall \mathcal{I}_1, \mathcal{I}_2 \in M$,

$$d(\mathbf{T}\mathcal{I}_1, \mathbf{T}\mathcal{I}_2) \le s d(\mathcal{I}_1, \mathcal{I}_2). \tag{2}$$

The construction of attractor coding scheme is governed by the Banach fixed point theorem [4]:

Theorem 1 *Let (M, d) be a complete metric space and $\mathbf{T} : M \to M$ is a contraction, $\exists! \mathcal{I}_0 \in M$ s.t. $\mathbf{T}\mathcal{I}_0 = \mathcal{I}_0$. Moreover, $\mathcal{I}_0$ can be found by*

$$\mathcal{I}_0 = \lim_{n\to\infty} \mathbf{T}^n \mathcal{I} \tag{3}$$

for any $\mathcal{I} \in M$ and $\mathbf{T}^{\circ n}$ denotes the n-th iteration of $\mathbf{T}$. $\mathcal{I}_0$ is called the fixed point of $\mathbf{T}$.

Given an image $\mathcal{I}_0 \in M$, the idea of attractor coding is to formulate a contraction $\mathbf{T} : M \rightarrow M$ those fixed point is $\mathcal{I}_0$. One way to construct $\mathbf{T}$ is the partitioned iterated function systems (PIFS) coding.

3. PARTITIONED ITERATED FUNCTION SYSTEMS

An image $\mathcal{I}_0$ of $2^N \times 2^N$ pixels is partitioned into a number of square blocks of size $2^{B+1} \times 2^{B+1}$ called domain blocks $\mathcal{D}_{i,j}$ where (i,j) denotes the pixel coordinates of the upper-left corner of the block. Let $D \equiv \{\mathcal{D}_{i,j} : i = 2^B p, j = 2^B q, 0 \leq p, q < 2^{N-B}\}$ be the collection of all domain blocks $\mathcal{D}_{i,j}$. After this, another partition into smaller blocks of size $2^B \times 2^B$ (range blocks $\mathcal{R}_{i,j}$) takes place and let $R \equiv \{\mathcal{R}_{i,j} : i = 2^B p, j = 2^B q, 0 \leq p, q < 2^{N-B}\}$ be the collection of all $\mathcal{R}_{i,j}$. For each $\mathcal{R}_{i,j}$ the affine transformation with the inclusion of orthogonalization operator [5] is constructed by the following way: define $\mathbf{P}_{i,j} : \mathcal{D}_{k,l} \rightarrow \mathcal{R}_{i,j}$ as the operator which gets a domain block $\mathcal{D}_{k,l}$ and places it at $\mathcal{R}_{i,j}$. Let $\tau_{i,j} : \mathcal{D}_{k,l} \rightarrow \mathcal{R}_{i,j}$ be the transformation which approximates each $\mathcal{R}_{i,j} \in R$ with a linear combination of orthogonalized $\mathcal{D}_{k,l} \equiv \mathbf{P}_{i,j}^{-1}(\mathcal{R}_{i,j})$ and a matrix $\mathbf{1}$ all whose elements are 1, i.e.,

$$\tau_{i,j}(\mathcal{D}_{k,l}) = s_{i,j} \times \mathbf{D}\mathbf{I}_{i,j}(\mathcal{D}_{k,l} - \overline{\mathcal{D}}_{k,l}) + o_{i,j} \times \mathbf{1} \tag{4}$$

where $\mathbf{D}$ and $\mathbf{I}_{i,j}$ are the decimation by 2 and the isometry which consists of one of eight different transformations of $\mathcal{R}_{i,j}$ by 90 degree rotations and reflections respectively. $\overline{\mathcal{D}}_{k,l}$ denotes the average value of $\mathcal{D}_{k,l}$. $s_{i,j}$ and $o_{i,j}$ are the scaling and offset scalars.

In the encoding procedure, for every range block $\mathcal{R}_{i,j}$ the best matching domain block is searched among all domain blocks such that $d(\ \mathcal{R}_{i,j},\ \tau_{i,j}\mathbf{P}_{i,j}^{-1}(\mathcal{R}_{i,j})\ )$ is minimized. Then, the image $\mathcal{I}_0$ can be approximated as follows:

$$\mathcal{I}_0 = \bigcup_{\mathcal{R}_{i,j} \in R} \mathcal{R}_{i,j} \approx \bigcup_{\mathcal{R}_{i,j} \in R} \tau_{i,j}\mathbf{P}_{i,j}^{-1}(\mathcal{R}_{i,j}) \equiv \mathbf{T}\mathcal{I}_0. \tag{5}$$

Compression is achieved by storing only the descriptions of the transformation $\mathbf{T}$. In the decoding procedure, the contractive transformation is performed recursively on any initial image. The fixed point of the transformation $\mathbf{T}$ is the desired image.

4. LAPPED PARTITIONED ITERATED FUNCTION SYSTEMS

In this work, a PIFS coding scheme with smooth range block lapping (LPIFS) is proposed. An image $\mathcal{I}_0$ is partitioned into a number of square range blocks of size $2^B \times 2^B$ in which every block overlaps each of its adjacent blocks with half of its support. The original range block collection R is replaced by $\tilde{R}$ in which the range blocks $\tilde{\mathcal{R}}_{i,j}$ satisfy the following conditions:

$$\begin{aligned} \tilde{\mathcal{R}}_{i,j} \bigcap \tilde{\mathcal{R}}_{k,l} &\neq \phi \quad \text{if they are neighbors,} \\ \tilde{\mathcal{R}}_{i,j} \bigcap \tilde{\mathcal{R}}_{k,l} &= \phi \quad \text{otherwise.} \end{aligned}$$

Except those pixels which lie on the boundary block, every pixel is covered by four range blocks with this lapped partition. Each region in $\mathcal{I}_0$ is now expressed as a weighted summation of four affine-transformed domain blocks. The weights $w_{i,j}$ are chosen in order to provide smooth overlapping across block boundaries in order to minimize the blocking artifacts. Let $\mathbf{W}$ be the matrix those entries are the weight $w_{i,j}$, i.e.,

$$\mathbf{W} = \begin{bmatrix} w_{0,0} & \cdots & w_{0,2^B-1} \\ \vdots & \ddots & \vdots \\ w_{2^B-1,0} & \cdots & w_{2^B-1,2^B-1} \end{bmatrix}. \tag{6}$$

The sum of $w_{i,j}$ for different adjacent range blocks on the same spatial location must be 1. That is

$$w_{i,j} + w_{i+2^{B-1},j} + w_{i,j+2^{B-1}} + w_{i+2^{B-1},j+2^{B-1}} = 1, \quad \forall\ 0 \leq i,j < 2^{B-1}. \tag{7}$$

For every weighted range block the best matching domain block is searched among all domain blocks such that the distance between the weighted range block and the transformed weighted domain block is minimized, i.e.,

$$\min_{\tau_{i,j},\mathbf{P}_{i,j}} d(\ \mathbf{W} * \tilde{\mathcal{R}}_{i,j},\ \mathbf{W} * (\tau_{i,j}\mathbf{P}_{i,j}^{-1}\tilde{\mathcal{R}}_{i,j})\ ) \tag{8}$$

for each $\mathcal{R}_{i,j}$. $*$ denotes the pointwise multiplication of two square matrices. Thus, the image can be approximated by

$$\begin{aligned} \mathcal{I}_0 &\equiv \{i_{u,v}^{(0)}\} \\ &= \bigcup_{i_{u,v}^{(0)} \in \mathcal{I}_0} [\sum_{\tilde{\mathcal{R}}_{i,j} \in \Omega_{u,v}} (\mathbf{W} * \tilde{\mathcal{R}}_{i,j})] \\ &\approx \bigcup_{i_{u,v}^{(0)} \in \mathcal{I}_0} [\sum_{\tilde{\mathcal{R}}_{i,j} \in \Omega_{u,v}} (\mathbf{W} * \tau_{i,j}\mathbf{P}_{i,j}^{-1}(\tilde{\mathcal{R}}_{i,j}))] \\ &\equiv \tilde{\mathbf{T}}\mathcal{I}_0 \end{aligned} \tag{9}$$

where $\Omega_{u,v}$ denotes the subset of $\tilde{R}$ that contains those $\tilde{\mathcal{R}}_{i,j}$ on which the pixel $i_{u,v}^{(0)}$ lies and $\tilde{\mathbf{T}}$ is our desired LPIFS. In the decoding procedure, the pixels in each iteration are computed as a weighted sum of the lapped adjacent range blocks with $w_{i,j}$ as their weights.

4.1. Choice of the Weighing Matrix W

If $w_{i,j}$ is chosen to be 1 for $\frac{1}{4}\times 2^B \leq i,j < \frac{3}{4}\times 2^B$ and 0 otherwise, it becomes the conventional PIFS. If all $w_{i,j}$ is chosen to be $\frac{1}{4}$, it becomes the overlapping PIFS proposed in [2]. Thus, the conventional PIFS and [2] can be seen as special cases of the proposed LPIFS.

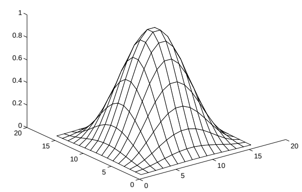

Figure 1: The weighing matrix $\mathbf{W}$ of dimension 16×16.

From (9), $\mathbf{W}$ is to filter the affined-transformed domain blocks and thus is analogue to the synthesis filter bank in subband coding. It was observed by Aase [6] that the synthesis filter should provide blocking-free reconstruction. The filter must be short enough to avoid excessive ringing effect and long enough for sufficient smoothness. The first basis of the lapped orthogonal transform (LOT) [7] is a good choice to fulfill the above requirements. Thus, $\mathbf{W}$ is constructed by using the linear combination of $\mathbf{1}$ and the first basis of LOT under the constraint (7). Our constructed $\mathbf{W}$ is shown in Fig. 1.

4.2. Aliasing Elimination

It was found that the proposed LPIFS blurs the image details and produces decoded image with larger error than that of conventional PIFS coding. The main cause of this problem is the aliasing error introduced in the weighting operation during decoding. Such blurring can be avoided if the global collage distance for all blocks is minimized simultaneously. However, a large system of equations will be involved. Thus, aliasing elimination is proposed to solve this problem. By doing so, each $\tilde{\mathcal{R}}_{i,j}$ can be processed independently.

For simplicity of presentation, we only describe the case where only two blocks overlap. Without loss of generality, consider the region $\tilde{\mathcal{R}}$ on which both $\tilde{\mathcal{R}}_i$ and $\tilde{\mathcal{R}}_{i+1}$ overlap, i.e., $\tilde{\mathcal{R}} \equiv \tilde{\mathcal{R}}_i \bigcap \tilde{\mathcal{R}}_{i+1}$. Let $\tau_i \mathbf{P}_i^{-1}(\tilde{\mathcal{R}}) - \tilde{\mathcal{R}} = E_1$ and $\tau_{i+1}\mathbf{P}_{i+1}^{-1}(\tilde{\mathcal{R}}) - \tilde{\mathcal{R}} = E_2$. Then, the error between the original and the decoded image on $\tilde{\mathcal{R}}$, $d(\tilde{\mathbf{T}}\tilde{\mathcal{R}}, \tilde{\mathcal{R}})$, can be expressed as

$$\begin{aligned}&\|\mathbf{W} * (\tau_i \mathbf{P}_i^{-1}(\tilde{\mathcal{R}}) - \tilde{\mathcal{R}}) \\ &\quad + (\mathbf{1}-\mathbf{W}) * (\tau_{i+1}\mathbf{P}_{i+1}^{-1}(\tilde{\mathcal{R}}) - \tilde{\mathcal{R}})\|\end{aligned}$$

Thus,

$$\begin{aligned} d(\tilde{\mathbf{T}}\tilde{\mathcal{R}}, \tilde{\mathcal{R}})^2 &= \|\mathbf{W} * E_1 + (\mathbf{1}-\mathbf{W}) * E_2\|^2 \\ &= \|\mathbf{W} * E_1\|^2 + \|(\mathbf{1}-\mathbf{W}) * E_2\|^2 \\ &\quad + 2 < \mathbf{W} * E_1, (\mathbf{1}-\mathbf{W}) * E_2 > \end{aligned} \tag{10}$$

The last term in (10) is the aliasing error which is introduced by the weighing operation and independent collage optimization of $\tilde{\mathcal{R}}_i$ and $\tilde{\mathcal{R}}_{i+1}$. After minimizing the collage distance for $\tilde{\mathcal{R}}_i$, the computation of the transformation parameters of $\tilde{\mathcal{R}}_{i+1}$ is modified by replacing $\tilde{\mathcal{R}}_{i+1}$ by $\tilde{\mathcal{R}}_{i+1} - (\mathbf{1}-\mathbf{W})^{-1} * \mathbf{W} * E_1$ where the inverse operator of $(\mathbf{1}-\mathbf{W})$ is pointwise.

4.3. Contractivity of LPIFS

The proposed LPIFS needs to be a contraction so that the decoded image is unique and can be found by iterative transformation. The proof of the contractivity of the proposed LPIFS is follows:

The proposed LPIFS can be seen as the weighted summation of four PIFSs, $\mathbf{T}_m, m = 0,1,2,3$, on an image. All $\mathbf{T}_m$ have the same $\tau_{i,j}$ in (4) and range block size but different corresponding range block partitions $R_m, m = 0,1,2,3$ defined as follows:

$$\begin{aligned} R_1 &\equiv \{\mathcal{R}_{i,j} : i = 2^B p, j = 2^B q\} \\ R_2 &\equiv \{\mathcal{R}_{i,j} : i = 2^B p + 2^{B-1}, j = 2^B q\} \\ R_3 &\equiv \{\mathcal{R}_{i,j} : i = 2^B p, j = 2^B q + 2^{B-1}\} \\ R_4 &\equiv \{\mathcal{R}_{i,j} : i = 2^B p + 2^{B-1}, j = 2^B q + 2^{B-1}\} \end{aligned}$$

where p, q are non-negative integers. Then, the contractivity of the proposed LPIFS is governed by proposition 1:

Lemma 1 *Let $\mathbf{T}_1$, $\mathbf{T}_2$ be the contractions on (M,d) with the same fixed point $\mathcal{I}_0 \in (M,d)$. $\forall \omega_1, \omega_2 \in \mathbb{R}^+$ s.t. $\omega_1 + \omega_2 = 1$, define $\mathbf{T}\mathcal{I} \equiv \omega_1 \mathbf{T}_1 \mathcal{I} + \omega_2 \mathbf{T}_2 \mathcal{I}$. Then, $\mathbf{T}$ must be a contraction and has the same fixed point $\mathcal{I}_0$.*

Proof: *Let $\mathcal{I}_1, \mathcal{I}_2 \in (M,d)$,*

$$\begin{aligned} d(\mathbf{T}\mathcal{I}_1, \mathbf{T}\mathcal{I}_2) &= \|\mathbf{T}(\mathcal{I}_1 - \mathcal{I}_2)\| \\ &= \|\omega_1 \mathbf{T}_1(\mathcal{I}_1 - \mathcal{I}_2) + \omega_2 \mathbf{T}_2(\mathcal{I}_1 - \mathcal{I}_2)\| \\ &\leq \|\omega_1 \mathbf{T}_1(\mathcal{I}_1 - \mathcal{I}_2)\| + \|\omega_2 \mathbf{T}_2(\mathcal{I}_1 - \mathcal{I}_2)\| \\ &= \omega_1\|\mathbf{T}_1(\mathcal{I}_1 - \mathcal{I}_2)\| + \omega_2\|\mathbf{T}_2(\mathcal{I}_1 - \mathcal{I}_2)\| \\ &\leq \omega_1 s_1 \|\mathcal{I}_1 - \mathcal{I}_2\| + \omega_2 s_2 \|\mathcal{I}_1 - \mathcal{I}_2\| \\ &= s\|\mathcal{I}_1 - \mathcal{I}_2\| \end{aligned}$$

where s_1, s_2 are the contractivity of $\mathbf{T}_1$, $\mathbf{T}_2$ respectively and $s \equiv \omega_1 s_1 + \omega_2 s_2$. s must be less than 1 as $s_1, s_2 < 1$. $\mathcal{I}_0$ is the fixed point of $\mathbf{T}$ since $\mathbf{T}\mathcal{I}_0 = \omega_1 \mathbf{T}_1 \mathcal{I}_0 + \omega_2 \mathbf{T}_2 \mathcal{I}_0 = \omega_1 \mathcal{I}_0 + \omega_2 \mathcal{I}_0 = \mathcal{I}_0$.

Proposition 1 *Let* $\mathbf{T}_m, m = 1, 2, ..., N$ *be the contractions on* (M, d) *with the same fixed point* $\mathcal{I}_0$. $\forall \omega_m \in \mathbb{R}^+, m = 1, 2, ..., N$ *s.t.* $\sum_m \omega_m = 1$, $\tilde{\mathbf{T}} \equiv \sum_m \omega_m \mathbf{T}_m$ *must be a contraction with fixed point* $\mathcal{I}_0$.
Proof: *Prove by mathematical induction with the use of lemma 1.*

5. EXPERIMENTAL RESULTS

Experiments have been carried out on several images. Each of the images is partitioned into 16×16 overlapping range blocks. The parameters of the transformation are quantized and coded in the same way as in [8]. It is found that LPIFS results higher PSNRs than PIFS at the same bit rates for most images (Table 1) and better visual quality for all images tested. Fig. 2 shows the zoom-in view of the decoded images from both the conventional PIFS and LPIFS for the image Lenna. The conventional PIFS has visible blocking effect which is significantly reduced in the image by LPIFS. The proposed LPIFS provides smooth transition between adjacent blocks by lapped blocks. The image details are preserved by using the aliasing elimination.

Table 1: Performance of the proposed scheme

Image	Proposed LPIFS		Conventional PIFS	
	bpp	PSNR	bpp	PSNR
Baboon	0.40	20.52	0.45	20.65
Flower	0.41	33.96	0.42	32.56
Fruits	0.41	32.08	0.42	31.74
Girl	0.41	37.74	0.42	36.25
Harbour	0.41	25.09	0.42	24.82
Lenna	0.41	32.03	0.42	31.01
Tiffany	0.41	28.62	0.42	29.86

6. REFERENCES

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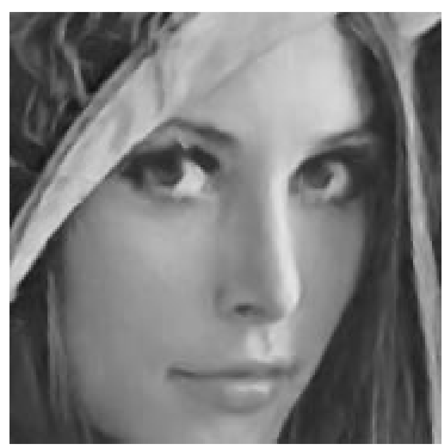
(a) PIFS: 0.42 bpp, PSNR=31.01dB



(b) LPIFS: 0.41 bpp, PSNR=32.03dB

Figure 2: Zoom-in view of decoded image Lenna.