A New Decoding Algorithm for Fractal Image Compression

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Abstract: A new iterative decoding method is proposed for fractal image compression. Convergence properties are provided. Experimental results show the superiority of the new method over the conventional decoding procedure.

1 Introduction

In conventional fractal coding, the image $x^* \in \mathbf{R}^n$ is stored as a unique fixed point x_T of an affine mapping T defined by $T(x) = Ax + b$, where A is a real $n \times n$ matrix and b is a real column vector of order n. The reconstruction of the image by the decoder proceeds by iteratively applying T to any arbitrary initial image x_0. If the spectral radius $\rho(A)$ of the matrix A is less than one, then the sequence of iterates $\{T^{(k)}(x_0)\}_k$ converges to $x_T = (I_n - A)^{-1}b$, where I_n is the identity matrix of order n. Here a vector $T^{(k)}(x)$ which we also denote by $x^{(k)}$ is defined by the recurrence relation

$$x^{(k+1)} = T(x^{(k)}) = Ax^{(k)} + b. \tag{1}$$

In this letter, we propose an improvement of the decoding procedure (1) in the spirit of the Gauss-Seidel method.

2 The new algorithm

For clarity, we explain our technique for the simple fractal scheme suggested by Monro and Dudbridge [1] where the image is segmented into N_D nonoverlapping domain blocks, each domain block $D_i, i \in \{1, \ldots, N_D\}$ being made up of an integer number m of adjacent range blocks $R_{ij} \in \mathbf{R}^{n_R}$, $j \in \{1, \ldots, m\}$, and where the same domain block D_i is used for the encoding of the m range blocks R_{ij}. In the encoding, for each range block R_{ij}, a scaling

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factor s_{ij} and offset o_{ij} are determined such that $\|R_{ij} - (s_{ij}\bar{D}_i + o_{ij}\mathbf{1})\|$ is minimum in the least squares sense. Here $\bar{D}_i$ is the block obtained after reducing the size of the domain block D_i to the size of a range block by the averaging of m adjacent pixels, and $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^{n_R}$. It can be easily seen that the matrix $A = (a_{u,v})$ is then a block diagonal matrix consisting of N_D square submatrices A_i, i.e., $A = diag(A_i)$, such that

$$A_i = \frac{1}{m} \begin{pmatrix} \mathbf{s}_{i1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_{i1} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{s}_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{s}_{im} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_{im} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{s}_{im} \end{pmatrix},$$

where $\mathbf{s}_{ij} = (s_{ij}, \ldots, s_{ij})$, and $\mathbf{0} = (0, \ldots, 0) \in \mathbf{R}^m$. In [2], it has been shown that with only minor restrictions

$$\rho(A) = \max_i \rho(A_i) = \max_i \frac{1}{m} |\sum_{j=1}^{m} s_{ij}|$$

which explains why the decoding process (1) may converge even though the absolute value of some of the scaling factors is larger than 1. However, for a fast encoding, the scaling factor of a given range block has to be fixed independently from the scaling factors of the other range blocks. Thus, it is common use in fractal coding to restrict the value of the scaling factors to the interval $(-1, 1)$. In the following, this assumption will also be taken.

We introduce now the new iteration technique. Let the k^{th} iterate $x^{(k)}$ of the new method be expressed as $x^{(k)} = (D_1^{(k)}, \ldots, D_{N_D}^{(k)})$, where $D_i^{(k)} = (R_{i1}^{(k)}, \ldots, R_{im}^{(k)})$. Let J denote a subset of $\{1, \ldots, N_D\}$. Then, whenever $i \in J$, we keep updating $D_i^{(k)}$ in the determination of $R_{ij+1}^{(k+1)} = s_{ij+1}\bar{D}_i^{(k)} + o_{ij+1}\mathbf{1}$ by using the latest estimates ${R_{i1}}^{(k+1)}, \ldots, {R_{ij}}^{(k+1)}$. If $i \notin J$, then $R_{ij+1}^{(k+1)}$ is computed as in the conventional scheme. Let each matrix A_i be expressed as the matrix sum $A_i = L_i + S_i$, where for $i \in J$, the matrix L_i is strictly lower triangular defined by ${l^i}_{u,v} = 0$ if $(p-1)n_R + 1 \le u \le pn_R$ and $v \ge (p-1)n_R + 1$, $p \in \{1, \ldots, m\}$, and ${l^i}_{u,v} = {a^i}_{u,v}$ otherwise, and where for $i \notin J$, L_i is the null matrix. Then the new scheme corresponds to the iterative method $x^{(k+1)} = Lx^{(k+1)} + Sx^{(k)} + b$, or

$$x^{(k+1)} = (I_n - L)^{-1} S x^{(k)} + (I_n - L)^{-1} b, \tag{2}$$

where $L = diag(L_i)$, and $S = diag(S_i)$. It is not difficult to prove that the proposed method, which can be seen as a special Gauss-Seidel method, satisfies:
(i) The iterative method (2) converges to x_T, the limit vector of the method (1), for any starting vector x_0.
(ii) Method (2) requires more arithmetic operations than method (1), precisely $m(m-1)|J|$, *only* at the first iteration.

Since the spectral radius of an iteration matrix is a valuable indicator of the rate of convergence of the corresponding iterative method [3], it is reasonable to choose J so that if $i \in J$, then

$$\rho((I_{mn_R} - L_i)^{-1} S_i) < \rho(A_i). \tag{3}$$

For example, using results on regular splitting of matrices [3], one can prove that inequality (3) is satisfied for all i such that $0 < s_{ij} < 1$ for all $j \in \{1, \ldots, m\}$ or $-1 < s_{ij} < 0$ for all $j \in \{1, \ldots, m\}$. Yet our method was efficient even when the updating was done unconditionally, i.e., after each computation of a new range block. In this case $J = \{1, \ldots, N_D\}$.

Our algorithm has a straightforward generalization to more complex fractal schemes where for a given range a search through a pool of domains is allowed. We still use the most recently computed ranges to update the domains involved in the decoding. Details and proofs of results can be found in [4].

3 Experimental results

Figure 1 shows the PSNR as a function of the number of decoding steps, starting from an initial black image for the conventional decoding, and the new method where the updating was done unconditionally. The image was encoded with Fisher's quadtree code [5] with the default parameters (three-level quadtree, one-class search). Clearly our new decoding scheme converged faster than the conventional one. The same observation held for all other tested images.

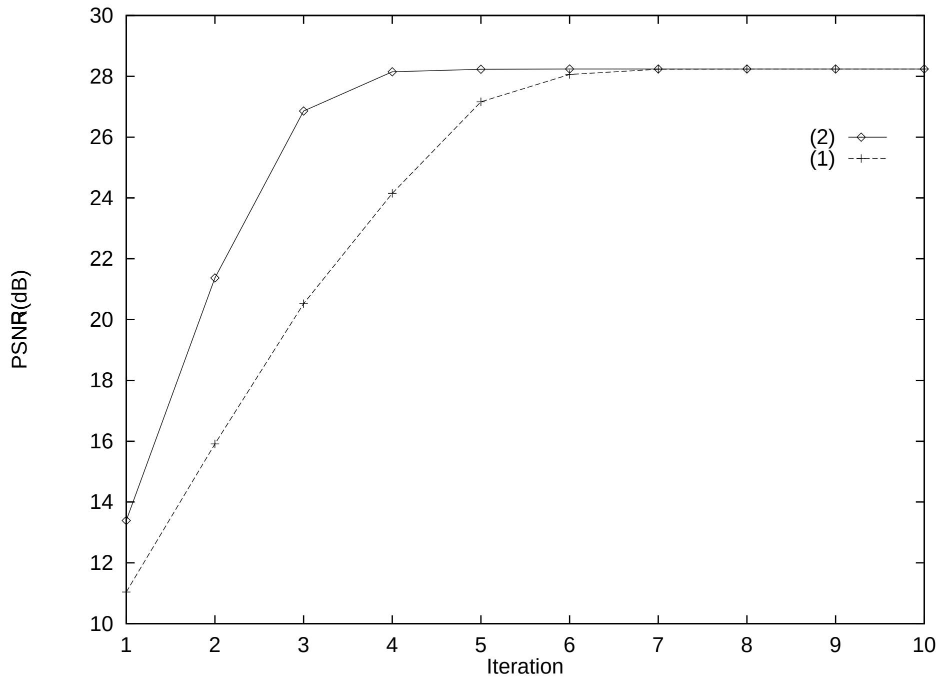

Figure 1: PSNR vs. iteration step for the 512 $\times$ 512 Lena image. (1): conventional decoding. (2): new method.

4 Conclusion

We have introduced a fast decoding technique for fractal image compression. To the best of our knowledge the only other existing schemes for fast decoding are the hierarchical based decoders described, for example, in [5]. These techniques are not competitors. Merging the schemes holds the promise of yielding an even faster decoder.

References

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