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# Fractal Image Coding with a Multiscaling-Domain

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## SUMMARY

In order to improve the approximation capability of fractal coding, a fractal coding method using multiple domains is proposed in which an approximation is carried out by superposition of contracted images from multiple domain blocks. By combining the multidomain method with the multiscaling procedure of dividing blocks into regions and individually providing the scaling coefficients of brightness, a contraction transform with even better degrees of approximation is obtained and edge reconstruction is improved. As a result, excellent reconstructed images are obtained even for complicated images for which high reproducibility cannot be realized with the conventional fractal coding. Also, it is possible to reproduce contour regions containing edges. © 2003 Wiley Periodicals, Inc. Electron Comm Jpn Pt 3, 87(2): 79–87, 2004; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/ecjc.10081

**Key words:** fractal coding; image compression; self-similarity; multidomain; multiscaling.

## 1. Introduction

Due to its high compression capability, fractal coding which realizes image compression by taking advantage of the partial self-similarity of natural images has been studied extensively in recent years. Fractal coding has been proposed by Barnsley as an inverse problem of the Iterated Function System (IFS), a system to generate binary fractal drawings [1, 2], and has been extended to grayscale images by Jacquin [3, 4]. Many recent investigations have been carried out to improve coding performance [5–7]. However, no performance robust to practices such as applications to recording and storage media has been realized.

In fractal coding, the image is compressed by using the partial self-similarity of the image as redundancy. Self-similarity well approximating the block to be coded is extracted from the image and the transform parameter for a contraction transformation representing the self-similarity is used as code. Since structural properties over blocks are utilized, this method has the characteristics that block distortion is not very conspicuous. However, since natural images do not possess rigorous fractal properties, this contraction transform is an approximate one. Therefore, the coding characteristics of fractal coding depend strongly on the self-similarity of the image. In the case of images with complicated structures, it is in general difficult to find contraction transforms which have sufficient degrees of approximation. Even if many bits are assigned, it is difficult to obtain high reproducibility. In the present paper, a fractal coding method based on multiple domains is proposed. In this method, in contrast to the conventional methods in which a block is approximated by a single contraction transform, several domain blocks are extracted and individual compression transforms are applied to them. The block is approximated by combining them. In this way, finding a contraction transform with a better degree of approximation becomes possible. Also, a multiscaling-domain method has been proposed in which the multiscaling scheme [8] is combined with the multidomain method. In multiscaling, a block is divided into regions and scaling coefficients for brightness are individually assigned so that contraction transforms with even higher degrees of approximation are possible. At the same time, the objects within the image become more clearly defined. Therefore, in addition to improvement of the degrees of approximation in the compression transform, clear edge reproduction becomes possible.

In this paper, the multidomain method and the multiscaling method as well as their combination, the multiscaling-domain method, are explained briefly in Section 2. In Section 3, these methods are actually utilized for carrying out coding configuration. Simulation results and observations are presented in Section 4. The effectiveness of the proposed method is discussed in Section 5.

## 2. Multidomain Fractal Coding

### 2.1. Fundamental principle

In general, natural images have local approximate self-similarity. Therefore, a portion of the image can be approximated by compressing another portion. Using this property, fractal coding divides the image into blocks and seeks a compression transform that provides optimum approximations for each block. The transform parameters are then used as the codes.

The regeneration error in the fractal coding is guaranteed by the Collage Theorem as shown below [2]. Let the original image be $\mu_{org}$, and the reproduced image be $\nu$, and let $\mu_1$ and $\mu_2$ be arbitrary images. Then, if the transform $\lambda(\cdot)$ satisfies the compression expressed by

$$d(\lambda(\mu_1), \lambda(\mu_2)) < s \cdot d(\mu_1, \mu_2), \; 0 \leq s < 1 \quad (1)$$

then the error between the original image and the reproduced image is regulated by the following inequality according to the Collage Theorem:

$$d(\mu_{org}, \nu) \leq \frac{1}{1-s} d(\mu_{org}, \lambda(\mu_{org})) \quad (2)$$

where $d(\cdot, \cdot)$ denotes the distance between two images, $d(\mu_{org}, \nu)$ is the error between the original and reproduced images, and $d(\mu_{org}, \lambda(\mu_{org}))$ is the approximation error due to the compression transform. This implies that convergence is guaranteed to a reproduced image well approximating the original image if a transform providing sufficient compression and sufficient similarity is given. Better reproduced images can be obtained if transforms with higher degrees of approximation are given. Hence, image reproduction by fractal coding depends on how effectively a compression transform with high approximation accuracy can be found.

In conventional fractal coding, the image is divided into nonoverlapping range blocks $R_{ij}$. A domain block $D_{kl}$ with a larger size corresponding to each range block is sought and then the transform $\lambda_{ij}(\cdot)$ providing the optimum approximation is derived (see Fig. 1). This transform $\lambda_{ij}(\cdot)$ is represented by the compression affine transform and is described by the following:

$$\begin{aligned} R'_{ij}(x,y) &= R_{ij}(x,y) - R_{ave} \\ D'_{ij}(x,y) &= \varepsilon\big(s\big(D_{kl}(x,y)\big) - D_{ave}\big) \\ R'_{ij}(x,y) &\cong \alpha \cdot D'_{ld}(x,y) \end{aligned} \quad (3)$$

where $R'_{ij}$ and $D'_{kl}$ are the range block and domain block with their average values separated; $R_{ave}$ is the average of the range blocks, $D_{ave}$ is the average of the domain blocks, $\varepsilon(\cdot)$ is the equal length transform, $s(\cdot)$ is the compression transform, and $\alpha$ is the scaling factor. By means of the least-squares method, $\alpha$ can be derived from the expression

$$\alpha = \frac{\sum_{x,y \in Block} R'_{ij}(x,y) \cdot D'_{kl}(x,y)}{\sum_{x,y \in Block} D'_{kl}(x,y)^2} \quad (4)$$

Decoding is realized by repeatedly applying the compression transform from the domain block to the range block. First, an arbitrary initial image divided into range blocks $R_{ij}$ is prepared. Subsequently, domain blocks $D_{kl}$ corresponding to each $R_{ij}$ are extracted. By means of the following transform (5), they are replaced by $R_{ij}$:

$$R_{ij} = \alpha \cdot \varepsilon\big(s\big(D_{kl}(x,y)\big) - D_{ave}\big) + R_{ave} \quad (5)$$

The above process is applied to all range blocks; this is considered as the first decoding process. The compression transform is again applied repeatedly to the decoded image. In this way, the gray level of the image converges and a reproduced image can be obtained.

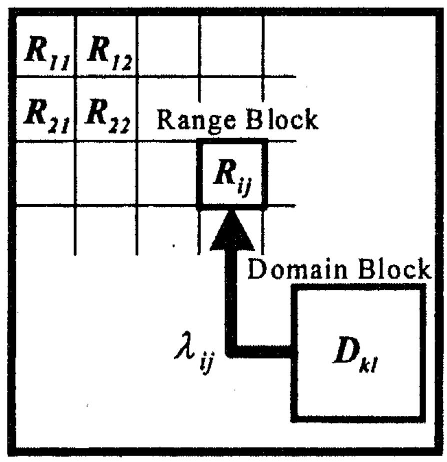



Fig. 1. Transformation from domain to range.

### 2.2. Multidomain method

The coding characteristics of the fractal coding depend strongly on the fractal nature of the image. Therefore, it is difficult to seek a compression transform with a sufficient degree of approximation for images with weak self-similarity or with violent gray-level changes. As a result, good image reproduction is difficult. Hence, the present authors propose the multidomain method in which the range blocks are approximated by superposition of the compression images from several domain blocks. In the multidomain method, the approximation error, which is not quite approximated by the compression transform from the domain blocks, is approximated by compression transforms from several domain blocks. In this way, it becomes possible to provide a compression transform with a higher degree of approximation than the conventional method. Better reproduced images should be obtainable from the images for which good reproduction cannot be obtained by the conventional method.

The basic principle of the multidomain method is to extract several domain blocks while seeking the compression transform and then to approximate the range block by a combination of these compression transforms. We wish to enhance the degree of approximation of the compression transform. First, in the search for the domain blocks, several domain blocks are extracted. Each domain block is extracted from a preset position relative to the central domain block. After the same compression transform and the equal length transform are applied to each domain block, average value separation is applied and then orthogonalization is carried out by the Gram-Schmidt orthogonalization method in order to prevent overlap of the components among the domains. Orthogonalization of the domain blocks is processed according to the preset sequence. After orthogonalization, the expansion coefficients $\alpha_m$ are derived in such a way that an optimum approximation is obtained:

$$\alpha_1 = \frac{\sum_{x,y\epsilon Block} R'_{ij}(x,y)\cdot D^{\dagger}_{1,kl}(x,y)}{\sum_{x,y\epsilon Block} D^{\dagger}_{1,kl}(x,y)^2}$$

$$\alpha_m = \frac{\sum_{x,y\epsilon Block}\left(R'_{ij}(x,y) - \sum_{n=1}^{m-1}\alpha_n\cdot D^{\dagger}_{n,kl}(x,y)\right)\cdot D^{\dagger}_{m,kl}(x,y)}{\sum_{x,y\epsilon Block} D^{\dagger}_{m,kl}(x,y)^2}$$

$$(m > 1) \qquad (6)$$

Hence, Eq. (3) can be written as

$$R'_{ij}(x,y) \cong \sum_{m-1}^{M} \alpha_m \cdot D^{\dagger}_{m.kl}(x,y) \qquad (7)$$

(see Fig. 2). Here, $D^{\dagger}_{m,kl}$ is the domain block to which orthogonalization is applied, and $M$ is the number of domain blocks to be superposed. By adjusting the number of domain blocks, it is possible to adjust the degree of approximation.

In the multidomain method, the average value of the block $R_{ave}$, the expansion coefficients $\alpha_{1\sim M}$, the equal length transform $\varepsilon$, and the position of the domain block $(k, l)$ are used as the codes.

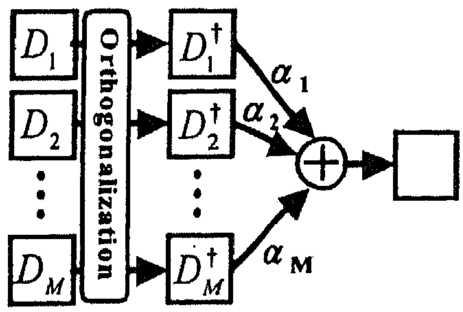



Fig. 2. Coding with multidomain.

### 2.3. Multiscaling method

The multiscaling method [8] is a method of dividing the range block into several regions by block truncation coding (BTC) [9] and providing a compression transform with a higher degree of approximation with different scaling values. In the present section, the principle of the multiscaling method and its applications to fractal coding are described. The number of regions in the range block is taken to be 2.

Coding using the multiscaling method is carried out as follows. First, the area within the block is divided into high-level and low-level regions. Then the average of the gray levels in the block is derived. The pixels that exceed the average are assigned to the high-level region and those below the average are assigned to the low-level region. The high-level regions are then assigned a value of 1 and the low-level regions a value of 0 so that a bit plane $B_{ij}$ is formed:

$$B_{ij}(x,y) = \begin{cases} 1, & R_{ij}(x,y) \geq R_{ave} \\ 0, & R_{ij}(x,y) < R_{ave} \end{cases} \qquad (8)$$

Further, for the domain block $D'_{kl}$ to which the compression transform and the equal length transform have been applied, a bit plane that divides high-level regions and low-level regions with the same shape as the range block is adopted. For each region, average value separation is performed and the scaling coefficients $\alpha_H$ and $\alpha_L$ for the high-level and low-level regions are derived by the following equations, obtained by the least-squares method:

$$\alpha_H = \frac{\sum_{x,y \in High} R'_{ij}(x,y) \cdot D'_{kl}(x,y)}{\sum_{x,y \in High} D'_{kl}(x,y)^2} \tag{9}$$

$$\alpha_L = \frac{\sum_{x,y \in Low} R'_{ij}(x,y) \cdot D'_{kl}(x,y)}{\sum_{x,y \in Low} D'_{kl}(x,y)^2} \tag{10}$$

By using $\alpha_H$ and $\alpha_L$, Eq. (3) can be rewritten as

$$R'_{ij}(x,y) \cong \begin{cases} \alpha_H \cdot D'_{kl}(x,y), B_{ij}(x,y) = 1 \\ \alpha_L \cdot D'_{kl}(x,y), B_{ij}(x,y) = 0 \end{cases} \tag{11}$$

In the multiscaling method, the average values $R_{Have}$ and $R_{Lave}$ of the block, the scaling coefficients $\alpha_H$ and $\alpha_L$, the equal length transform ε, the position $(k, l)$ of the domain block, and the bit plane $B_{ij}$ are used as the codes. Many bits are required for representation of the bit plane. By vector quantization treating the bit plane as a binary vector, the bit rate of the bit plane can be reduced to 0.4 to 0.5 bpp.

### 2.4. Multiscaling-domain method

Both the multidomain and multiscaling methods are ways to improve the degree of approximation of the compression transform in fractal coding. By combining the multidomain and multiscaling methods, further improvement of the approximation is expected.

Coding by the multiscaling-domain method is performed as follows. First, the range block is divided into high-level regions and low-level regions. Several domain blocks are extracted and subjected to the same compression transform and equal length transform. Subsequently, average value separation is applied on a region-by-region basis to each domain block. According to the preset sequence, orthogonalization is carried out by the Gram-Schmidt method. Then, the optimum expansion coefficients $\alpha_{1\sim M}$ for each domain block are derived by means of Eq. (6). For the blocks approximated by the multidomain method, the bit plane is assigned and approximation by multiscaling is performed. The scaling coefficients $\alpha_H$ and $\alpha_L$ for each region are obtained by the following equations based on the least-squares method:

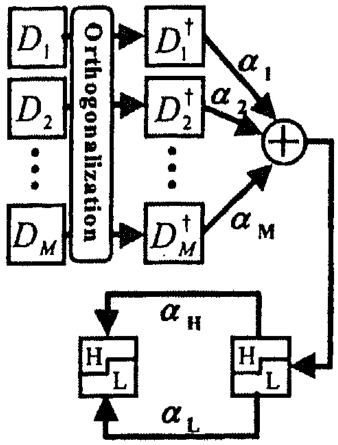



Fig. 3. Coding with multiscaling-domain.

$$\alpha_H = \frac{\sum_{x,y \in High} \left( R'_{ij}(x,y) \cdot \sum_{m=1}^{M} \alpha_m \cdot D^\dagger_{m,kl}(x,y) \right)}{\sum_{x,y \in High} \left( \sum_{m=1}^{M} \alpha_m \cdot D^\dagger_{m,kl}(x,y) \right)^2} \tag{12}$$

$$\alpha_L = \frac{\sum_{x,y \in Low} \left( R'_{ij}(x,y) \cdot \sum_{m=1}^{M} \alpha_m \cdot D^\dagger_{m,kl}(x,y) \right)}{\sum_{x,y \in Low} \left( \sum_{m=1}^{M} \alpha_m \cdot D^\dagger_{m,kl}(x,y) \right)^2} \tag{13}$$

By using these expansion coefficients $\alpha_{1\sim M}$ and the two scaling coefficients $\alpha_H$ and $\alpha_L$, Eq. (13) can be rewritten as

$$R'_{ij}(x,y) \cong \begin{cases} \alpha_H \cdot \sum_{m=1}^{M} \alpha_m \cdot D^\dagger_{m,kl}(x,y) \\ B_{ij}(x,y) = 1 \\ \alpha_L \cdot \sum_{m=1}^{M} \alpha_m \cdot D^\dagger_{m,kl}(x,y) \\ B_{ij}(x,y) = 0 \end{cases} \tag{14}$$

(see Fig. 3). In the multiscaling-domain, the average values $R_{Have}$ and $R_{Lave}$ of the block, the scaling coefficients $\alpha_{1\sim M}$, $\alpha_H$, $\alpha_L$, the equal length transform ε, the location $(k, l)$ of the domain block, and the bit plane $B_{ij}$ are used as the codes.

## 3. Coding Configuration

In the previous section a coding procedure using the multidomain and the multiscaling was described. In the present section, a coding configuration for the application of each method to actual coding is explained. The multiscaling method is intended to increase the degree of approximation by division of regions to enhance the reproducibility of edges. Therefore, it is considered effective to apply multiscaling to the range blocks in which strong contrast is contained. Hence, multiscaling is applied to the blocks with high variance after the block variances $\sigma^2$ are derived. On the other hand, the multidomain method is applied to the cases where a sufficient degree of approximation is not realized by coding in a single domain.

The coding composition is shown in Fig. 4. First, for block discrimination, the image is divided into range blocks $R_{ij}$ and the variance $\sigma^2$ of all blocks is derived. By means of the preset threshold $Z_b$, the blocks with values less than $Z_b$ are judged to be shade blocks and those with values greater than $Z_b$ to be edge blocks. The blocks judged to be shade have low variances so that the interior of the block can be approximated sufficiently by using only the average value. Hence, only the average values are used as the codes. No search is conducted for self-similarity. Hence, the coding time and the bit rate can be reduced. On the other hand, the blocks judged to be edge blocks are divided into two at a constant rate according to the variance $\sigma^2$. The group with a lower $\sigma^2$ is considered as a single scaling block and self-similarity is sought by the conventional compression transform. The blocks with higher $\sigma^2$ are considered to be multiscaling blocks, for which self-similarity is searched by multiscaling. The edge blocks are coded once in the single domain and the approximation error is derived. With reference to the preset threshold value $Z_E$, the single domain is still used for coding if the mean-square error is lower than $Z_E$. On the other hand, if the error is larger than $Z_E$, then a transform with a higher degree of approximation is derived by the multidomain method. If the edge block is a single-scaling block, the multidomain method is used, but the multiscaling-domain method is used if it is multiscaling.

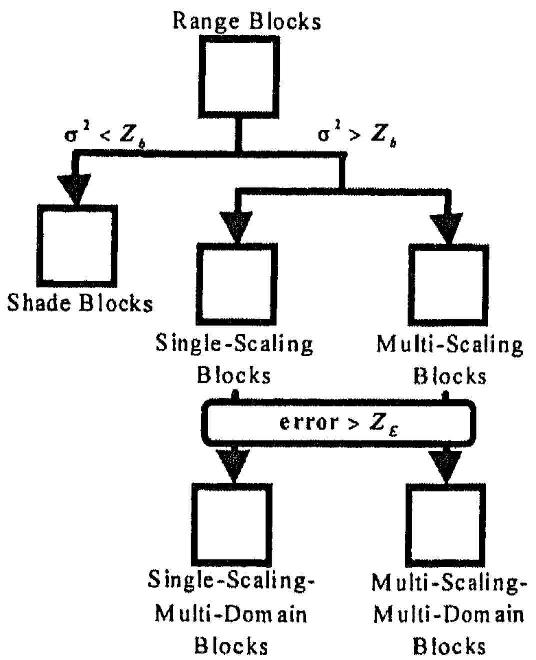
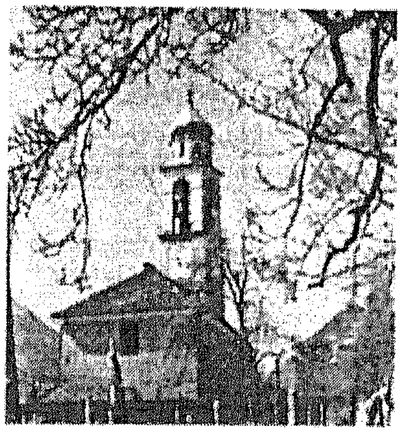



Fig. 4. Coding composition.

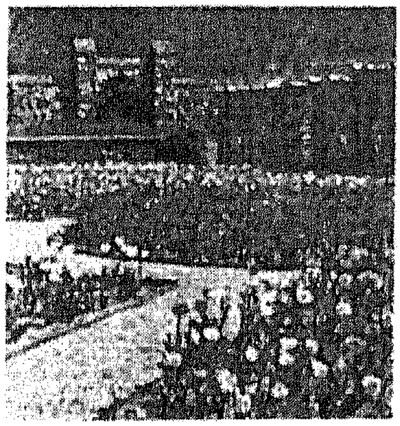
## 4. Simulation Results

By using "Church," "Girl," and "Tulip" (256 x 256 pels, 8 bits in Fig. 5), coding simulation is performed



(a) Church

(b) Girl



(c) Tulip

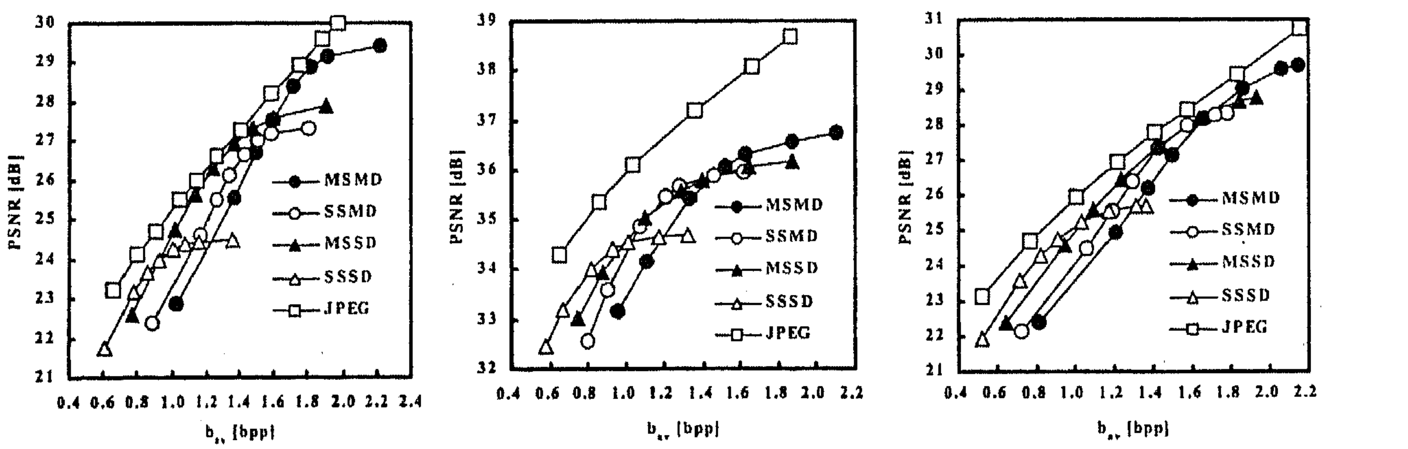
Fig. 5. Original image.

Table 1. Setting of coding

| | Domain | $\alpha$ | $\varepsilon$ | $R_{ave}$ | MS:SS | $M$ | $Z_E$ |
|---|---|---|---|---|---|---|---|
| SSSD | 10 | 4 [bits] | 3 [bits] | 5 [bits] | - | 1 | - |
| MSSD | 10 | 4×2 | 3 [bits] | 5 [bits] | 1:1 | 1 | - |
| SSMD (Church) | 10 | 4×$M$ [bits] | 3 [bits] | 5×2 | - | 5 | 150 |
| SSMD (Girl) | 10 | 4×$M$ [bits] | 3 [bits] | 5 [bits] | - | 5 | 20 |
| MSMD (Church) | 10 | 4($M$+2) | 3 [bits] | 5×2 | 1:1 | 5 | 150 |
| MSMD (Girl) | 10 | 4($M$+2) | 3 [bits] | 5×2 | 1:1 | 5 | 20 |

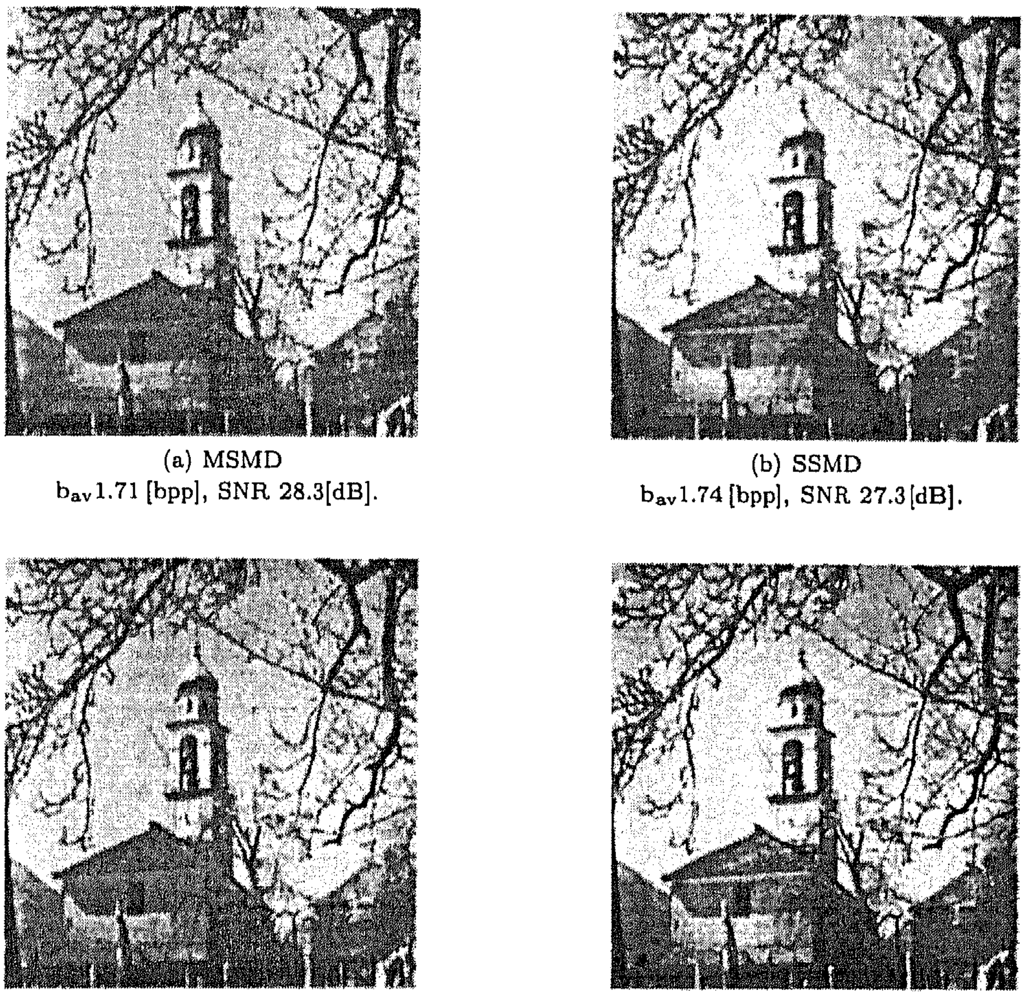
according to the coding composition presented in Section 3. The coding setup is shown in Table 1. Figure 6 shows the coding characteristics obtained by simulation and Fig. 7 shows the images reconstructed by each method. Let MSMD be the approach using the multiscaling-domain method according to Section 3, SSMD be the approach using the multidomain method considering all edge blocks as single-scaling blocks, MSSD be the approach using only the multiscaling method and not the multidomain method, and SSSD be the conventional method without the multiscaling or multidomain method. The bit rate is adjusted by varying the rates of the shade blocks and edge blocks. In fractal coding, the PSNR is saturated even if the bit rate is increased due to the limitation of the degree of approximation by the compression transform. For "Church," the value of the PSNR is improved by 4.9 dB by MSMD and by 2.8 dB by SSMD in comparison with SSSD. In comparison with MSSD, the improvement is 1.5 dB by the MSMD, so that reconstruction images with better accuracy than the conventional method can be obtained. Similar characteristics are also obtained for "Girl" and "Tulip." The PSNR for "Girl" is improved by 2.0 dB compared with SSSD and by 0.5 dB compared with MSSD. For "Tulip," the improvement is 4.0 dB compared with SSSD and 0.9 dB compared with MSSD. In SSMD, the reproducibility of the image is improved more by the use of the multidomain method than by the single-domain method. However, sufficient performance is not obtained in comparison with MSSD. In the case of "Church," there are many locations with strong contrast such as tree branches, and it is more effective to use the multiscaling method with two levels at each block. On the other hand, in "Girl," with few blocks that have strong variance, there are places where SSMD provides better performance. In comparison with JPEG, rather similar coding performance is obtained in the images containing fine details such as in Tulip. Coding performance is poorer for images with many flat locations such as those in Girl.

Figure 8 shows the percentage of the block type at each bit rate for SSMD and MSSD. Since the amount of coding needed for the multidomain method is greater than that required for the multiscaling method, the percentage of shade blocks is higher in relative terms at the same bit rate. Also, to reduce the bit rate, it is necessary to set $Z_b$ to a large value. As $Z_b$ becomes larger, the percentage of blocks with




(a) Church (b) Girl (c) Tulip

Fig. 6. Coding property.



(a) MSMD
$b_{av}$1.71 [bpp], SNR 28.3[dB].

(b) SSMD
$b_{av}$1.74 [bpp], SNR 27.3[dB].

(c) MSSD
$b_{av}$1.72 [bpp], SNR 27.8[dB].

(d) JPEG
$b_{av}$1.71 [bpp], SNR 28.7[dB].

Fig. 7. Decoded image.

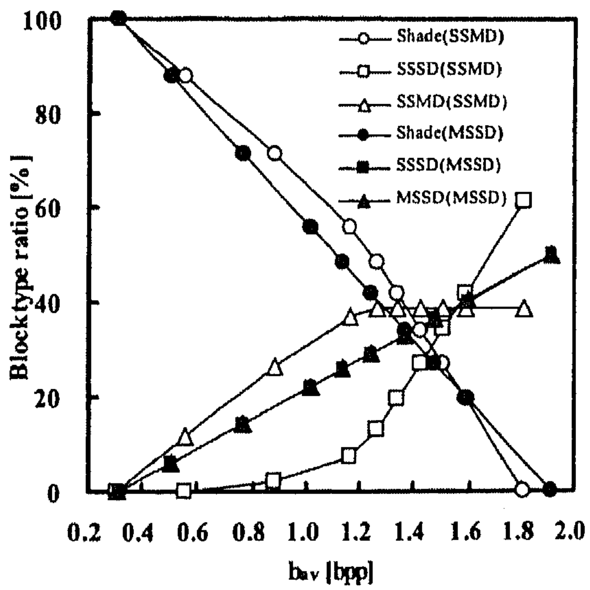
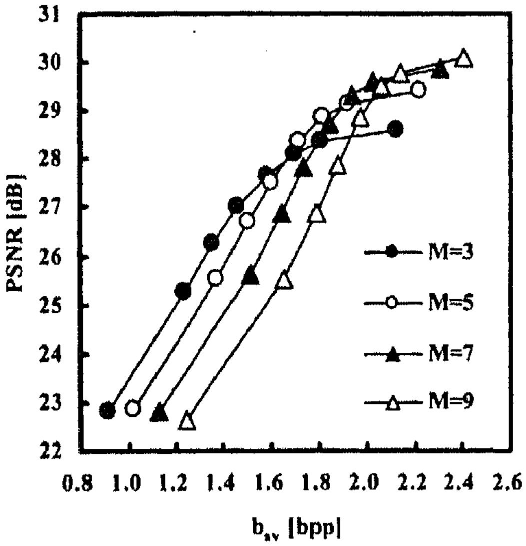


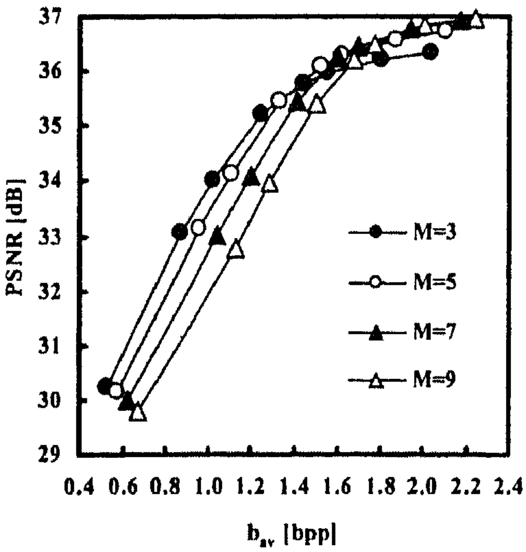

Fig. 8. Relation between bit rate and block-type ratio.

greater variance is larger in the edge blocks, so that the percentage of multidomain blocks in all entire edge blocks is higher. Therefore, in the present procedure for discriminating multiple domains, the coding performance is seen to decrease.

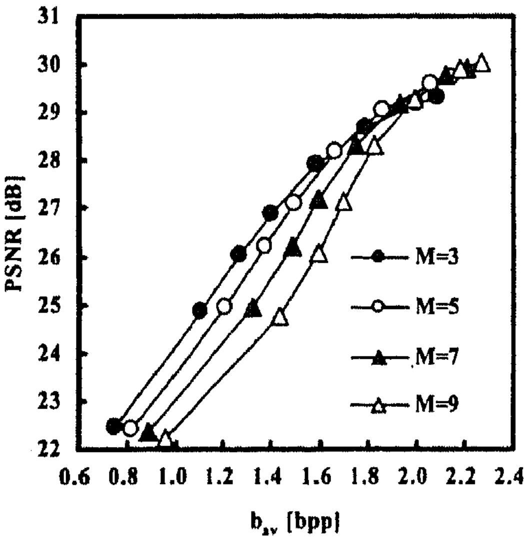
Next, the coding characteristics of MSMD are studied as the number of domains is varied. The coding setting is shown in the MSMD column. The results are shown in Fig. 9. As the number of overlapping domain blocks $M$ is increased, the saturation value of PSNR is seen to increase. However, in general, as the approximation error of the compression transform becomes smaller, the improvement of the degree of approximation due to the addition of new domains becomes smaller. Hence, as the number of domains is increased, the improvement of the saturated value becomes smaller. When more domains are used, the corresponding increase in the number of bits becomes enormous and is not realistic. The number of domains, which is fixed




(a) Church




(b) Girl




(c) Tulip

Fig. 9. Relation between number of domains and coding property.

at $M = 5$ in the present case, can be adaptively changed according to the degree of approximation required for improvement.

## 5. Conclusions

In this paper, to deal with the problem that in fractal approximation, high reproducibility becomes troublesome for images without sufficient fractal nature, a multidomain method is proposed that allows a higher degree of approximation than the conventional method by combining the compression transforms from several domain blocks. In addition, the multiscaling-domain method—a combination of the multidomain method and the multiscaling method—is proposed. Making use of these methods, a high PSNR was obtained for images for which sufficient approximation was not realized by the conventional method. In the case of "Church," the saturation value of the PSNR was increased by 4.9 dB. The increase was found to be 1.5 dB in comparison with the multiscaling method. However, since many bits are needed in the multidomain blocks, the PSNR is shown to be depressed at low bit rates.

For improvement of coding performance at a low bit rate, it is necessary to determine adaptively the number of overlapping blocks in multiple domains. In the present case, the number of domains in the multidomain method is fixed. It is possible to increase the number of domains from a single domain until a sufficient degree of approximation is achieved, depending on the difficulty of the blocks. As the block size is increased, the bit rate is decreased. Although it is difficult to obtain a sufficient degree of approximation, it is nonetheless possible to obtain a contraction transform with a high degree of approximation by the multiscaling-domain method. Hence, there exists an open question to be investigated for the block size.

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