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SOLVING THE INVERSE PROBLEM FOR FUNCTION/IMAGE APPROXIMATION USING ITERATED FUNCTION SYSTEMS II. ALGORITHM AND COMPUTATIONS

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In this paper, we provide an algorithm for the construction of IFSM approximations to a target set $v \in \mathcal{L}^2_+(X,\mu)$, where $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$ (Lebesgue measure). The algorithm minimizes the squared "collage distance" $\| v - Tv \|_2^2$. We work with an infinite set of *fixed* affine IFS maps $w_i : X \to X$ satisfying a certain density and nonoverlapping condition. As such, only an optimization over the grey level maps $\phi_i : \mathbf{R}^+ \to \mathbf{R}^+$ is required. If affine maps are assumed, i.e. $\phi_i = \alpha_i t + \beta_i$, then the algorithm becomes a quadratic programming (QP) problem in the α_i and β_i. We can also define a "local IFSM" (LIFSM) which considers the actions of contractive maps w_i on *subsets* of X to produce smaller subsets. Again, affine ϕ_i maps are used, resulting in a QP problem. Some approximations of functions on [0,1] and images in $[0,1]^2$ are presented.

1. INTRODUCTION

This paper represents a continuation of Paper I presented at this conference.[1] As such, the same notation is employed.

2. THE INVERSE PROBLEM IN $\mathcal{L}^2(X,\mu)$ AS A QUADRATIC PROGRAMMING PROBLEM

We now describe an algorithm for the construction of IFSM approximations of arbitrary accuracy to a target set $v \in \mathcal{L}^2(X,\mu)$. Because of our primary interest in the problem of image representation, our discussion is restricted to the approximation of nonnegative *image functions*, $u : X \to \mathbf{R}^+ = [0,\infty)$, i.e. $v \in \mathcal{L}^2_+(X,\mu) = \{f \in \mathcal{L}^2(X,\mu) | f(x) \geq 0, \forall x \in X\}$. (There is no loss of generality in this assumption since the grey-level map parameters β_i can be used to shift the range of functions $u : X \to \mathbf{R}$ without affecting the contractivity of the T operator.) For an N-map contractive IFSM $(\mathbf{w},\Phi)$ on (X,d) with associated operator T, the squared $\mathcal{L}^2$ collage distance is given by

$$\begin{aligned}\Delta^2 &= \| v - Tv \|_2^2 \\ &= \int_X [\sum_{k=1}^{N}{}' \phi_k(v(w_k^{-1}(x))) - v(x)]^2 d\mu. \qquad (1)\end{aligned}$$

Following our discussion in Paper I, we consider the IFS maps w_i to be fixed. The problem reduces to the determination of grey level maps ϕ_i which minimize the collage distance Δ^2. In the special case that

1. $\cup_{k=1}^N X_k = \cup_{k=1}^N \hat{w}_i(X) = X$, i.e. the sets X_k "tile" X, and
2. $\mu(\hat{w}_i(X) \cap \hat{w}_j(X)) = 0$ for $i \neq j$,

then the squared collage distance Δ^2 becomes

$$\begin{aligned}\Delta^2 &= \sum_{k=1}^{N} \int_{X_k} [\phi_k(v(w_k^{-1}(x)) - v(x)]^2 d\mu \\ &= \sum_{k=1}^{N} \Delta_k^2, \qquad (2)\end{aligned}$$

i.e. the sum of collage distances over the nonoverlapping subsets X_k. The minimization of each integral is a continuous version of "least squares", with respect to the measure μ: For each subset X_k, find the $\phi_k : \mathbf{R}^+ \to \mathbf{R}^+$ which provides the best $\mathcal{L}^2(X,\mu)$ approximation to the graph of $v(x)$ vs. $(v \circ w_k^{-1})(x)$ for $x \in X$.

Most, if not all, applications in the literature assume the nonoverlapping property due to its simplicity, in addition to the assumptions that $\mu = m^{(D)}$ and that the w_k are similitudes. In addition, the grey level maps are assumed to be affine, i.e. $\phi_i(t) = \alpha_i t + \beta_i$. The standard approach is to impose stationarity conditions on Δ_i^2 which yield a set of linear equations in α_i and β_i. There is no guarantee, however, that the solutions to these equations will be nonnegative, ensuring that $\phi_i : \mathbf{R}^+ \to \mathbf{R}^+$. This may not be a great problem in actual applications, since $\phi_i(v(x))$ may, in fact, be nonnegative (or only very slightly negative) for practical images. Nevertheless, this detail is generally overlooked in the literature.

In this discussion, we shall not assume the nonoverlapping property: i.e. the sets $\hat{w}_i(X)$ are allowed to overlap on sets of nonzero measure. In what follows, we make the following assumptions:

1. $w_i \in Con(X)$ (but not necessarily in $Sim(X)$) and $\cup_{k=1}^N X_k = X$,
2. $c_k > 0$ for $1 \leq k \leq N$, and

3. $\phi_i : \mathbf{R}^+ \to \mathbf{R}^+$, where $\phi_i(t) = \alpha_i t + \beta_i$, $t \in \mathbf{R}^+$ (affine grey level maps). Thus, $\alpha_i, \beta_i \geq 0$ for $1 \leq i \leq N$.

The squared $\mathcal{L}^2$ collage distance then becomes

$$\begin{aligned} \Delta^2 &= < v - Tv, v - Tv > \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N}[< \psi_k, \psi_l > \alpha_k\alpha_l \ + \ 2 < \psi_k, \chi_l > \alpha_k\beta_l + \ < \chi_k, \chi_l > \beta_k\beta_l] \\ &\quad - 2\sum_{k=1}^{N}[< v, \psi_k > \alpha_k + < v, \chi_k > \beta_k] + < v, v >, \end{aligned} \tag{3}$$

where

$$\psi_k(x) = u(w_k^{-1}(x)), \quad \chi_k(x) = I_{w_k(X)}(x), \quad x \in X. \tag{4}$$

Note that Δ^2 is a quadratic form in the ϕ-map parameters α_i and β_i, i.e.

$$\Delta^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c, \tag{5}$$

where $\mathbf{x}^T = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N}$. The elements of the symmetric matrix $\mathbf{A}$ are given by

$$a_{i,j} =< \psi_i, \psi_j >, \quad a_{N+i,N+j} =< \chi_i, \chi_j >, \quad a_{i,N+j} =< \psi_i, \chi_j >, \quad 1 \leq i, j \leq N. \tag{6}$$

As well,

$$b_i = -2 < v, \psi_i >, \quad b_{N+i} = -2 < v, \chi_i >, \quad 1 \leq i \leq N, \tag{7}$$

and $c =< v, v >=\| v \|_2^2$.

The minimization of Δ^2 is a quadratic programming (QP) problem in the parameters α_i and β_i, $i = 1, 2, ..., N$, i.e.

$$\text{minimize} \quad \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c, \quad \mathbf{x} \geq 0. \tag{8}$$

In order to guarantee that a minimum of this quadratic form exists on a compact set of feasible parameters α_i, β_i, we impose the additional condition

$$\| Tv \|_1 \ \leq \ \| v \|_1 . \tag{9}$$

In terms of the grey level map parameters, this is a linear inequality constraint, i.e.

$$\sum_{k=1}^{N}(\alpha_k \| v \circ w_k^{-1} \|_1 + \beta_k\mu(X_k)) \ \leq \ \| v \|_1 . \tag{10}$$

For the special case $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$, which will be used in all applications, the above linear inequality constraint becomes

$$\sum_{k=1}^{N} c_k^D(\alpha_k \| v \|_1 + V_X\beta_k) \ \leq \ \| v \|_1, \tag{11}$$

where $V_X = m^{(D)}(X)$.

For a given target $v \in \mathcal{L}^p(X, \mu)$, assuming $\| v \|_1 \neq 0$, we denote the feasible set of N-map IFSM grey-level parameters as

$$\Pi_v^{2N} = \{(\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N} \ : \| Tv \|_1 \leq \| v \|_1, \alpha_i, \beta_i \geq 0\}. \tag{12}$$

Note that Π_v^{2N}, which is compact in the natural topology on $\mathbf{R}^{2N}$, depends on the target function v.

Proposition 1 *Let $X \subset \mathbf{R}^D$, $\mu = m^{(D)}$ and $\mathbf{x}^T = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \Pi_v^{2N}$. Then the corresponding operator T is contractive in $(\mathcal{L}^1(X,\mu), d_1)$, except possibly when $\beta_1 = \beta_2 = \ldots = \beta_N = 0$.*

We now describe our algorithm. As before, let $\mathbf{w}$ be an infinite set of fixed affine contraction maps on $X \subset \mathbf{R}^D$ which generates a μ-dense and nonoverlapping family of subsets of X. Let

$$\mathbf{w}^N = \{w_1, w_2, ..., w_N\}, \quad N = 1, 2, \ldots \tag{13}$$

denote N-map truncations of $\mathbf{w}$. Given a target function $v \in \mathcal{L}^p(X, m^{(D)})$, the region Π_v^{2N} defined in Eq. (12) contains all feasible points $\mathbf{x}^N = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N}$, each of which defines a unique N-vector of affine grey level maps Φ^N,

$$\Phi^N = \{\alpha_1 t + \beta_1, ..., \alpha_N t + \beta_N\}. \tag{14}$$

For an $\mathbf{x}^N \in \Pi_v^{2N}$, let

$$T^N : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu) \tag{15}$$

be the operator associated with the N-map IFSM $(\mathbf{w}^N, \Phi^N)$ and let

$$\Delta_N^2 = \| v - T^N v \|_2^2 \tag{16}$$

denote the corresponding squared $\mathcal{L}^2$ collage distance. Since $\Delta_N^2 : \Pi_v^{2N} \to \mathbf{R}$ is a quadratic form, it is continuous in the natural topology on $\mathbf{R}^{2N}$ and thus attains a minimum value $\Delta_{N,min}^2$ on Π_v^{2N}. The following result ensures that $\Delta_{N,min}^2$ may be made arbitrarily small by this minimization procedure, thus providing a solution to the inverse problem.

Theorem 1 $\Delta_{N,min}^2 \to 0$ *as* $N \to \infty$.

The proof of this theorem is based on the proof of Theorem 6 in Paper I.

The advantages of QP have been discussed in our paper on solving the inverse problem for measures.[2] Briefly, (i) QP locates a minimum of Δ_N^2 on the simplex Π^{2N} in a finite number of steps and (ii) in many problems, the minimum is achieved on a boundary point of the simplex. In such cases, if $(\alpha_k, \beta_k) = (0,0)$ then $\phi_k(t) = 0$, which implies that the associated IFS map w_k is superfluous. QP (as opposed to gradient-type schemes) will locate such boundary points in a finite number of steps, essentially discarding such superfluous maps. The elimination of such maps represents an increase in the *data compression factor*. Some numerical calculations involving the QP method will be presented in the next section.

3. THE INVERSE PROBLEM WITH "LOCAL IFSM"

Our method can easily be generalized to incorporate the strategy of Jacquin,[3] namely, that we consider the actions of contractive maps w_i on *subsets* of X (the "parent blocks") to produce smaller subsets of X (the "child blocks"). This is also referred to as a "Local IFS".[4] Rather than trying to approximate a target as a union of contracted copies of itself as in the IFS method, the local IFS method approximates the target as a union of copies of *subsets* of itself. A good discussion of this method can be found in the review by Fisher,[5] who has also been involved in more detailed investigations.[6]

We should mention that formal solutions of the inverse problem using various Local IFSM can be formulated, in analogy to Theorem 6 of Paper I. As a result, one has theorems analogous to Theorem 1 of this paper, guaranteeing that the optimization method can provide solutions of arbitary accuracy. We omit a discussion of these results in this report.

3.1 A Simple Nonoverlapping Local IFSM

It is convenient to first formulate a simple "local IFSM" (LIFSM) on $\mathcal{L}^p(X,\mu)$, where $\mu = m^{(D)}$ as follows. Let $J_k \subset X$, $k = 1, 2, ..., N$, with $N \geq 1$, such that

1. $\cup_{k=1}^N J_k = X$ (tiling condition) and
2. $\mu(J_j \cap J_k) = 0$ for $j \neq k$ (μ-nonoverlapping condition).

In addition, suppose that for each $J_k, 1 \leq k \leq N$, there exists an $I_{j(k)} \subseteq X$ and a map $w_{i(k),k} \in Con(X)$, with contractivity factor $c_{i(k),k} > 0$, such that $w_{i(k),k}(I_{i(k)}) = J_k$. In other words, for each "child block" J_k, there is a corresponding "parent block" $I_{i(k)}$.

For each map $w_{i(k)} : I_{i(k)} \to J_k$, let there be a grey level map $\phi_k : \mathbf{R}^+ \to \mathbf{R}^+$. The vectors $\mathbf{w}_{loc} = \{w_{i(1),1}, ..., w_{i(N),N}\}$ and Φ comprise an N-map LIFSM $(\mathbf{w}_{loc}, \Phi)$. Now define an associated operator $T_{loc} : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu)$ as follows: For $u \in \mathcal{L}^p(X,\mu)$,

$$(T_{loc}u)(x) = \phi_k(u(w_{i(k),k}^{-1}(x))), \quad \text{for } x \in J_k, \quad k = 1, 2, ..., N. \tag{17}$$

Proposition 2 *Let $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$. Let $(\mathbf{w}_{loc}, \Phi)$ be a local IFSM defined as above, with $w_i \in Sim(X)$, $\phi_k \in Lip(\mathbf{R}^+)$ for $1 \leq k \leq N$. Then for $u, v \in \mathcal{L}^p(X, m)$,*

$$d_p(T_{loc}u, T_{loc}v) \leq C(D,p) d_p(u,v), \tag{18}$$

where

$$C(D,p) = [\sum_{k=1}^N c_{i(k),k}^D K_k^p]^{1/p}. \tag{19}$$

Proof: For $u, v \in \mathcal{L}^p(X, m^{(D)})$,

$$\begin{aligned} \| T_{loc}u - T_{loc}v \|_p^p &= \sum_{k=1}^N \int_{J_k} |\phi_k(u(w_{i(k),k}^{-1}(x))) - \phi_k(v(w_{i(k),k}^{-1}(x)))|^p dx \\ &= \sum_{k=1}^N c_{i(k),k}^D \int_{I_k} |\phi_k(u(y)) - \phi_k(v(y))|^p dy \\ &\leq \sum_{k=1}^N c_{i(k),k}^D K_k^p \int_{I_k} |u(y) - v(y)|^p dy \\ &\leq [\sum_{k=1}^N c_{i(k),k}^D K_k^p] \| u - v \|_p^p . \end{aligned} \tag{20}$$

If $C(D,p) < 1$, then T_{loc} is contractive over the space $(\mathcal{L}^p(X, m^{(D)}), d_p)$ and possesses a unique fixed point $\overline{u}$. Moreover, for any $u \in \mathcal{L}^p(X,\mu)$, $d_p(T_{loc}^n u, \bar{u}) \to 0$ as $n \to \infty$.

Now let $X \subset \mathbf{R}^D$, $\mu = m^{(D)}$ and $v \in \mathcal{L}^p(X, m^{(D)})$ be a target set. Given an N-map LIFSM as defined above, the squared $\mathcal{L}^2$ collage distance is given by

$$\begin{aligned} \Delta^2 &= \| T_{loc}v - v \|_2^2 \\ &= \sum_{k=1}^N \int_{J_k} [\phi_i(v(w_{i(k),k}^{-1}(x)) - v(x)]^2 dx \\ &= \sum_{k=1}^N \Delta_k^2. \end{aligned} \tag{21}$$

Again, because the child blocks are conveniently nonoverlapping, the problem reduces to the minimization of each squared collage distance Δ_k^2 over the block J_k, a "least squares" determination of ϕ_k. In the special case that the ϕ_k maps are affine, the minimization of each Δ_k^2 is, as before, a quadratic programming problem in the two parameters α_k and β_k.

3.2 Local IFSM With "More Degrees of Freedom"

The Local IFSM discussed above represents only one of many possible ways in which parent blocks may be mapped to child blocks. Some additional possibilities, to each of which would correspond a particular T_{loc} operator, are listed below:

1. For a given child cell J_k, we may wish to consider more than one parent cell I_j at the same time.
2. It may be possible, and indeed desirable, to consider more than one affine mapping from a given parent I_i to a given child J_k. For example, on [0,1], we can consider both the orientation-preserving and orientation non-preserving maps (e.g. $w_{ik} = s_{ik}x + a_{ik}$, with $s_{ik} = 1$ and -1, respectively). In $[0,1]^2$, there are eight possible contraction maps from a larger parent square to a smaller child square and we may wish to employ some or all of them in our T_{loc} operator.
3. Combining (1) and (2) above.
4. Overlapping child cells.

Clearly, there are many possibilities. From a practical viewpoint, however, there are limitations. In this section, we formulate the inverse problem associated with (2) above. (The extension of this method to (3) above is rather straightforward.) Some numerical calculations using this strategy have been performed and will be reported in the next section.

For simplicity, we assume equipartitions of $X = [0,1]^D$ which produce regular parent and child blocks, i.e. squares, cubes. As well, we assume that the tiling and μ-nonoverlapping conditions of Sec. 3.1 are also satisfied by the child blocks J_k. Let $w_{i(k),k}^{(l)}, l = 1, 2, ..., M_D$ denote the set of all possible similitudes mapping a parent block $I_{i(k)}$ to a child block J_k, all having a common contraction factor $c_{i(k),k}$ ($M_1 = 2, M_2 = 8, ...$). Associated with each IFS map $w_{i(k),k}^{(l)}$ will be a grey level map $\phi_k^{(k)} \in Lip(\mathbf{R}^+)$. Then the operator $T_{loc} : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu)$ associated with such an LIFSM is given by

$$(T_{loc}u)(x) = \sum_{l=1}^{M_D} \phi_k^{(l)}(u([w_{i(k),k}^{(l)}]^{-1}(x))), \quad for \;\; x \in J_k, \quad k = 1, 2, ..., N. \tag{22}$$

Since the child blocks J_k are nonoverlapping, the squared $\mathcal{L}^2$ collage distance separates into a sum of collage distances over each child cell J_k, i.e.

$$\begin{aligned}\Delta^2 &= < T_{loc}v - v, T_{loc}v - v > \\ &= \sum_{k=1}^{N} \Delta_k^2.\end{aligned} \tag{23}$$

In the case of affine grey level maps, i.e.

$$\phi_k^{(l)} = \alpha_k^{(l)} t + \beta_k^{(l)}, \quad 1 \le l \le M_D, \tag{24}$$

each collage distance is given by

$$\Delta_k^2 = \sum_{l=1}^{M_D}\sum_{m=1}^{M_D}[< \psi_l, \psi_m > \alpha_k^{(l)}\alpha_k^{(m)} + 2 < \psi_l, \chi_m > \alpha_k^{(l)}\beta_k^{(m)} + < \chi_l, \chi_m > \beta_k^{(l)}\beta_k^{(m)}]$$
$$- 2\sum_{l=1}^{M_D}[< v, \psi_l > \alpha_k^{(l)} + < v, \chi_l > \beta_k^{(l)}] + < v_k, v_k >, \quad (25)$$

where

$$\psi_l(x) = u([w_{i(k),k}^{(l)}]^{-1}(x)), \quad \chi_l(x) = I_{J_k}(x), \quad v_k(x) = v(x)I_{J_k}(x). \quad (26)$$

At first sight, it would appear that each Δ_k^2 is a quadratic form in the $2M_D$ parameters $\alpha_k^{(l)}$ and $\beta_k^{(l)}$, $1 \le l \le M_D$. However, the functions $\chi_l(x)$ are identical. Therefore, Δ_k^2 reduces to the following quadratic form in the $M_D + 1$ parameters $\alpha_k^{(l)}$, $1 \le l \le M_D$ and $\beta_k = \sum_{l=1}^{M_D} \beta_k^{(l)}$:

$$\Delta^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c, \quad \mathbf{x} \ge 0, \quad (27)$$

where $\mathbf{x}^T = (\alpha_k^{(1)}, ..., \alpha_k^{(M_D)}, \beta_k) \in \mathbf{R}^{M_D+1}$. The elements of the symmetric matrix $\mathbf{A}$ are given by

$$a_{i,j} =< \psi_i, \psi_j >, \quad 1 \le i, j \le M_D \quad (28)$$

and

$$a_{i,M_D+1} = a_{M_D+1,i} =< \psi_i >, \quad 1 \le i \le M_D. \quad (29)$$

As well,

$$b_i = -2 < v, \psi_i >, \quad 1 \le i \le N, \quad (30)$$

and $b_{M_D+1} =< v_k >$, $c =< v_k, v_k >=\| v_k \|_2^2$.

The feasible set of parameters is chosen so that the following conditions are satisfied:

$$\Pi_{v_k}^{M_D+1} = \{(\alpha_k^{(1)}, ..., \alpha_k^{M_D}, \beta_k) \in \mathbf{R}^{M_D+1} | \; \| T_{loc}v_k \|_1 \le \| v_k \|_1, \alpha_k^{(l)}, \beta_k \ge 0\}. \quad (31)$$

4. APPLICATIONS AND NUMERICAL COMPUTATIONS

In this section we present some results of our algorithm to construct IFSM and LIFSM approximations to functions ($X = [0,1]$) and images ($X = [0,1]^2$). In all applications, $\mu = m^{(D)}$ is the Lebesgue measure.

4.1 Function Approximation on [0,1]

4.1.1 *Normal IFSM method*

Normal IFSM method, where the following "wavelet"-type basis of affine IFS maps has been employed,

$$w_{ij}(x) = 2^{-i}(x + j - 1), \quad i = 1, 2, ..., \quad j = 1, 2, ..., 2^i. \quad (32)$$

In the calculations, we consider N-map truncations $(\mathbf{w}^N, \Phi^N)$ and compute the minimum squared collage distance $\Delta_{N,min}^2$ in the feasible region Π_v^{2N} using a quadratic programming (QP) algorithm.

In Fig. 1 are shown some approximations to the target function $v(x) = \sin(\pi x)$. The target set as well as the attractors of the optimal truncated IFSM $(\mathbf{w}^N, \Phi^N)$ are plotted.

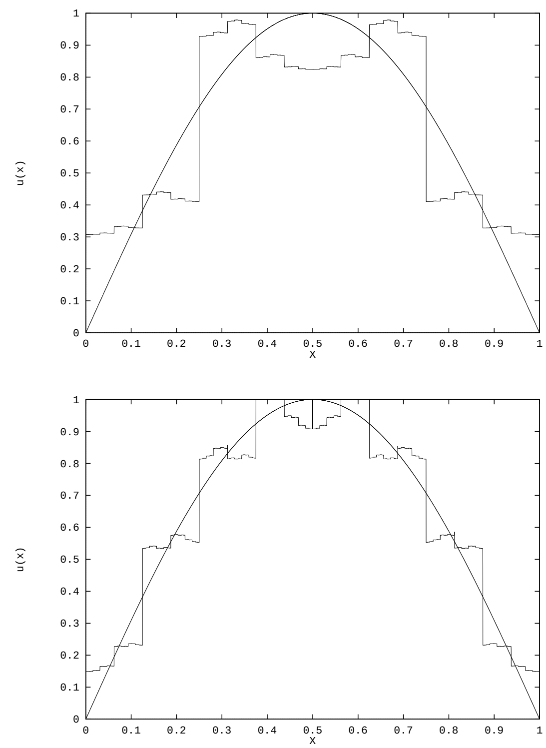

Fig. 1 Approximations to the target set $v(x) = sin(\pi x)$ on $X = [0, 1]$ yielded by the IFSM method of Sec. 2, using the wavelet-type basis of Eq. (32). In (a), $N = 6$ maps were used ($i = 2$ in Eq. (32)). In (b), $N = 14$ maps were used ($i = 3$).

In Fig. 1(a), $N = 6$ maps were used, i.e. $1 \leq i \leq 2$ in Eq. (32). In Fig. 1(b), $N = 14$ maps were used, i.e. $1 \leq i \leq 3$.

4.1.2 *Nonoverlapping Local IFSM method*

The child blocks J_k (as well as the parent blocks I_j) are simply the dyadic subintervals obtained by the action of the w_{ij} maps of Eq. (32) on [0,1]. As originally done by Jacquin,[3] for each child J_k we test a number of parents I_j. For each parent, in turn, we consider both possible affine contraction maps (i.e. orientation preserving and nonpreserving). We choose the parent and map which gives the minimum collage distance Δ_k^2. In Fig. 2 are shown some approximations to $v(x) = \sin(\pi x)$. In Fig. 2(a), we have used one parent block, $I_0 = [0, 1]$ and four child blocks $J_k = w_{2,k}(X)$. (This is identical to the "normal" IFSM method with four nonoverlapping IFS maps.) In Fig. 2(b), we have used two parent blocks, $I_j = w_{1,j}(X)$ and four child blocks $J_k = w_{2,k}(X)$. In Fig. 2(c), we use four parent blocks, $I_j = w_{2,j}(X)$, and eight child blocks, $J_k = w_{3,k}(X)$. It is not surprising that for a given number of IFSM maps, N, the local IFSM method yields much better results than normal IFSM, since the former seeks to tile the target $v(x)$ with only parts of itself.

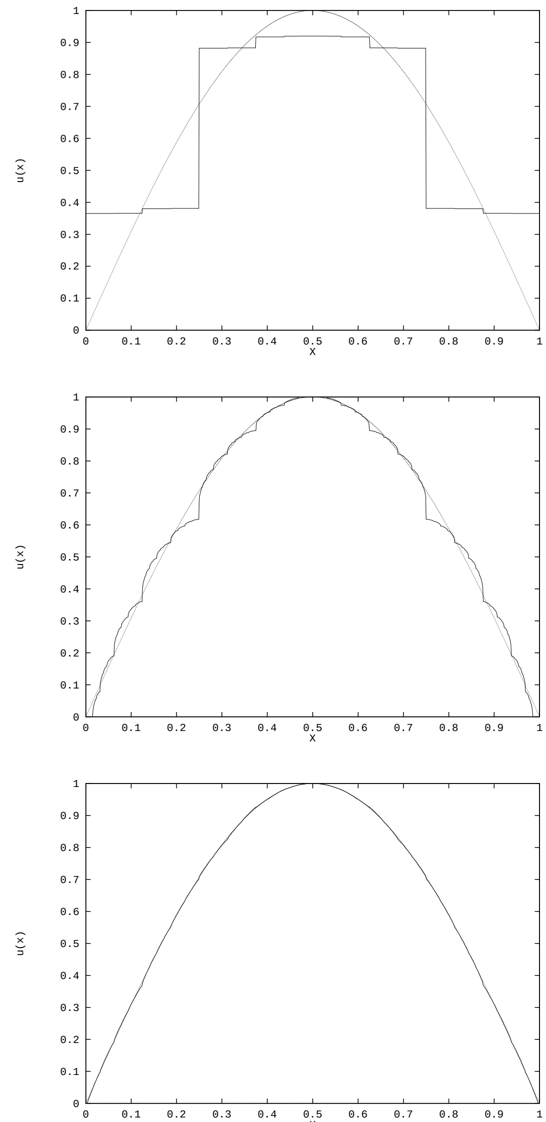

Fig. 2 Approximations to the target set $v(x) = sin(\pi x)$ on $X = [0,1]$ yielded by the "Local IFSM" method of Sec. 3.1 using n_1 parent cells and n_2 child cells. (a) $(n_1, n_2) = (1,4)$. (b) $(n_1, n_2) = (2,4)$. (c) $(n_1, n_2) = (4,8)$.

4.2 Image Approximation in $[0,1]^2$ Using Local IFSM

Figure 3 shows the target image "Lena", a 512×512 pixel greyscale image, with each pixel having 256 possible values (8 bits, with values from 0 to 255, which were rescaled to values in [0,1]).

4.2.1 *Nonoverlapping Local IFSM method*

Given a child block, J_k, we test all parent blocks I_j. For each parent block, we test all eight possible contraction maps $w_{i,j}^{(l)}, 1 \leq l \leq 8$, selecting the map which produces the minimum collage distance. Naturally, we then select the parent which produces the best overall collage of J_k.

Fig. 3 Target image $v(x,y)$: 512×512 pixel array, 8 bits per pixel.

In Fig. 4 we have used $8^2 = 64$ parent blocks and $16^2 = 256$ child blocks. The execution time was 296 s. (All computations were done in FORTRAN using an IBM Model 355 POWERStation equipped with a RISC processor. No special work was done to optimize the code. At this time, we were primarily interested in comparing execution times required for different IFSM methods.) In Fig. 5 we have used $16^2 = 256$ parent blocks and $32^2 = 1024$ child blocks. The execution time was 3383 s.

Fig. 4 Local IFSM approximation. $(n_1, n_2) = (8^2, 16^2)$. $\| \bar{u} - v \|_1 = 0.04$.

Fig. 5 Local IFSM approximation. $(n_1, n_2) = (16^2, 32^2)$. $\| \bar{u} - v \|_1 = 0.029$.

4.2.2 *Overlapping LIFSM method*

For each child block J_k, and each possible parent block I_j, consider all eight maps $w_{i,j}^{(l)}, 1 \leq l \leq 8$, simultaneously, i.e. we compute the minimum of the quadratic form in Eq. (27) - a quadratic programming problem in 9 unknowns. This approach actually represents a saving in time over the method used in Figs. 4 and 5. For example, the approximation in Fig. 6 was produced with 32^2 parents and 64^2 children, representing a two-fold increase in resolution over the approximation in Fig. 5.

Fig. 6 "Overlapping" Local IFSM approximation: 8 maps from parent I_j to J_k. $(n_1, n_2) = (32^2, 64^2)$. $\| \bar{u} - v \|_1 = 0.018$.

Nevertheless, the calculations required 3402 s of execution time, roughly the same time as required for Fig. 5. For almost all child cells J_k, the QP algorithm located a minimum of the collage distance Δ_k^2 on the boundary of $\Pi_{v_k}^9$, with no more than two grey level maps $\phi_k^{(l)}$ differing significantly from zero.

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