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Signal and Image Analysis

SOLVING THE INVERSE PROBLEM FOR FUNCTION/IMAGE APPROXIMATION USING ITERATED FUNCTION SYSTEMS I. THEORETICAL BASIS

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We are concerned with function approximation and image representation using Iterated Function Systems (IFS) over $\mathcal{L}^p(X,\mu)$: An N-map IFS with grey level maps (IFSM), to be denoted as $(\mathbf{w},\Phi)$, is a set $\mathbf{w}$ of N contraction maps $w_i : X \to X$ over a compact metric space (X,d) (the "base space") with an associated set Φ of maps $\phi_i : \mathbf{R} \to \mathbf{R}$. Associated with each IFSM is a contractive operator T with fixed point $\overline{u} \in \mathcal{L}^p(X,\mu)$. We provide a rigorous solution to the following inverse problem: Given a target $v \in \mathcal{L}^p(X,\mu)$ and an $\epsilon > 0$, find an IFSM whose attractor satisfies $\| \overline{u} - v \|_p < \epsilon$.

1. INTRODUCTION

The method of "Iterated Function Systems" (IFS)[1,2] has been very successful for the approximation of fractal sets and images. However, with differing applications and goals, IFS-type methods on various metric spaces have been devised, e.g. IFS with probabilities

(IFSP), Iterated Fuzzy Set Systems (IFZS) and IFS with grey level maps (IFSM). In these methods, the image or *target* is represented by either a probability distribution or a function in one or more spatial variables. All methods share the common feature that the target is represented by an element $\overline{y}$ of a given metric space (Y, d_Y). Such an element is then identified as the unique fixed point of a contraction map $T : Y \to Y$, i.e. $T\bar{y} = \bar{y}$. One may generate $\overline{y}$ by iterating T. In order to design an algorithm, one must necessarily define an appropriate space (Y, d_Y) and a contraction map T. We summarize below the various IFS-type methods in the chronological order in which they were introduced:

1.1 IFS[3,4]

On a compact metric space (X, d) (the "base space" of the IFS, typically a compact subset of $\mathbf{R}^n$, e.g. $[0,1], [0,1]^2$ with d the Euclidean distance on $\mathbf{R}^n$). Then $Y = \mathcal{H}(X)$, the set of nonempty compact subsets of X and d_Y is the Hausdorff metric h, defined as follows. Let the distance between a point $x \in X$ and a set $S \in \mathcal{H}(X)$ be given by

$$d(x, S) = \inf_{z \in S} d(x, z); \tag{1}$$

then for each $S_1, S_2 \in \mathcal{H}(X)$,

$$h(S_1, S_2) = \max\{\sup_{x \in S_1} d(x, S_2), \sup_{z \in S_2} d(z, S_1)\}. \tag{2}$$

Now let $\mathbf{w} = \{w_1, w_2, ..., w_N\}$, $w_i \in Con(X)$, where

$$Con(X) = \{w : X \to X | d(w(x), w(y)) \leq cd(x, y) \text{ for some } c \in [0, 1), \forall x, y \in X\}. \tag{3}$$

Associated with each contraction map w_i is a set-valued mapping $\hat{w}_i : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $\hat{w}_i(S) = \{w_i(x) : x \in S\}$ for $S \in \mathcal{H}(X)$. Then the operator T associated with the N-map IFS $\mathbf{w}$ is defined as follows:

$$T(S) = \bigcup_{i=1}^{N} \hat{w}_i(S), \quad S \in \mathcal{H}(X). \tag{4}$$

The contractivity of the maps w_i on (X, d) implies the contractivity of T on $(\mathcal{H}(X), h)$. The completeness[5] of $(\mathcal{H}(X), h)$ guarantees the existence of a unique fixed point $\bar{y} = A$ of T in $\mathcal{H}(X)$. The set A, also called the *attractor*, is the IFS representation of an image. From Eq. (4), it satisfies the following self-tiling property,

$$A = \bigcup_{i=1}^{N} \hat{w}_i(A). \tag{5}$$

1.2 IFSP[3,4]

To an N-map IFS $\mathbf{w}$ is now associated a set of probabilities $\mathbf{p} = \{p_1, p_2, ..., p_N\}$ with $\sum_{i=1}^{N} p_1 = 1$. Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X. Then $Y = \mathcal{M}(X)$, the set of all probability measures on $\mathcal{B}(X)$. Here, the (Markov) operator is defined as follows: For a $\nu \in \mathcal{M}(X)$ and each $S \in \mathcal{H}(X)$,

$$(T\nu)(S) = (M\nu)(S) = \sum_{i=1}^{N} p_i(\nu(\hat{w}_i^{-1}(S)). \tag{6}$$

The distance function d_Y on $Y = \mathcal{M}(X)$ is the so-called Hutchinson distance $d_H(\mu, \nu)$: For $\mu, \nu \in \mathcal{M}(X)$,

$$d_H(\mu, \nu) = \sup_{f \in Li_1(X)} \left[\int_X f d\mu - \int_X f d\nu \right], \tag{7}$$

where

$$Li_1(X) = \{f : X \to \mathbf{R} | |f(x_1) - f(x_2)| \leq d(x_1, x_2), \quad \forall x_1, x_2 \in X\}. \tag{8}$$

The contractivity of the IFS maps w_i implies the contractivity of the Markov operator M on $(\mathcal{M}(X), d_H)$. The completeness of $\mathcal{M}(X)$ guarantees the existence of a unique fixed point or *invariant measure* $\bar{\mu} \in \mathcal{M}(X)$ of M which has been used to represent the grey level distribution of a picture.

Some interesting and powerful variations of the IFSP have been developed, including (i) Recurrent IFS[6] and (ii) "Fractal Block Coding"[7] or "Local IFS".[8] We must also mention that the latter methods have been extended to function approximation (using affine IFS and grey level maps) primarily for the purpose of image compression.[9,10]

1.3 IFZS[11]

The image, including the grey levels of its pixels, is represented by a function $u : X \to [0, 1]$. The set Y is the family $\mathcal{F}^*(X)$ of all functions $u : X \to [0, 1]$ which are upper semicontinuous on (X, d) and such that $u(x_0) = 1$ for some $x_0 \in X$. We now define the metric $d_Y = d_\infty$ used for this case. First, let $u \in \mathcal{F}^*(X)$. Now, for each $\alpha \in [0, 1]$ define the α-level sets of u as follows:

$$\begin{aligned} [u]^\alpha &:= \{x \in X : u(x) \geq \alpha\}, \quad \alpha \in (0, 1], \\ [u]^0 &:= \overline{\{x \in X : u(x) > 0\}}, \end{aligned} \tag{9}$$

where $\bar{S}$ denotes the closure of the set S in (X, d). Clearly, $[u]^\alpha \in \mathcal{H}(X)$ for $0 \leq \alpha \leq 1$. Then for $u, v \in \mathcal{F}^*(X)$, define

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha). \tag{10}$$

Associated with a set of IFS contraction maps $\mathbf{w}$ is a set of functions $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$, $\phi_i : [0, 1] \to [0, 1]$, each of them being (i) nondecreasing and (ii) right continuous. In addition, (iii) $\phi_i(0) = 0$ for $1 \leq i \leq N$ and (iv) for at least one $i^* \in \{1, 2, ..., N\}$, $\phi_{i^*}(1) = 1$. Finally, the map $T = T_s : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$ is defined as

$$(Tu)(x) = (T_s u)(x) = \sup_{1 \leq i \leq N} \{\phi_i(\tilde{u}(w_i^{-1}(x))\}, \quad \forall x \in X, \tag{11}$$

where, for $B \subseteq X$, (i) $\tilde{u}(B) = \sup_{z \in B}\{u(z)\}$ if $B \neq \emptyset$ and (ii) $\tilde{u}(\emptyset) = 0$. The properties imposed on the functions ϕ_i along with the contractivity of the IFS maps w_i imply that T_s is a contraction map on $(\mathcal{F}^*(X), d_\infty)$. The completeness[12] of $(\mathcal{F}^*(X), d_\infty)$ guarantees the existence of a unique fixed point $\bar{u} \in \mathcal{F}^*(X)$ of the operator T_s. This function $\bar{u}$ will provide the grey level of each pixel x in the picture. The α-level sets of $\bar{u}$ obey the following generalized self-tiling property:

$$[\bar{u}]^\alpha = \bigcup_{i=1}^{N} w_i([\phi_i \circ \bar{u}]^\alpha), \quad \alpha \in [0, 1]. \tag{12}$$

The picture itself can be produced by iterating the map T_s, starting with any initial function $u_0 \in \mathcal{F}^*(X)$. The pair of vectors $(\mathbf{w}, \Phi)$ along with the associated map T_s defines an IFZS algorithm for image production.

Nevertheless, the Hausdorff metric d_∞ is very restrictive, from both practical (i.e. image processing) as well as theoretical perspectives. By making two fundamental modifications to the IFZS approach,[13] one arrives at an IFS with grey level maps over the space $\mathcal{L}^1(X, \mu)$. This, in turn, serves as the motivation to formulate IFS over the general function spaces $\mathcal{L}^p(X, \mu)$, to which we now turn our attention.

1.4 IFSM[13]

Let μ be a measure on $\mathcal{B}(X)$ and for any integer $p \geq 1$, let $\mathcal{L}^p(X, \mu)$ denote the linear space of all real–valued functions u such that u^p is integrable on $(\mathcal{B}(X), \mu)$. We choose $Y = \mathcal{L}^p(X, \mu)$. The distance function d_Y is defined by the usual L_p–norm, i.e.

$$d_Y(u, v) := d_p(u, v) = \left[\int_X |u(x) - v(x)|^p d\mu(x)\right]^{1/p}. \tag{13}$$

Associated with an N-map contractive IFS $\mathbf{w} = \{w_1, w_2, \ldots, w_N\}$ is a set of functions or *grey level maps* $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$, with $\phi_i : \mathbf{R} \to \mathbf{R}$. As suggested by the Markov operator in the IFSP algorithm, we define the operator T corresponding to the N-map IFSM $(\mathbf{w}, \Phi)$ as

$$(Tu)(x) := \sum_{k=1}^{N}{}' \phi_k(u(w_k^{-1}(x))). \tag{14}$$

The prime signifies that the sum operates on all those terms for which $w_k^{-1}(x)$ is defined. If $w_k^{-1}(x) = \emptyset$ for all $k = 1, 2, \ldots, N$, then $(Tu)(x) := 0$.

Under suitable restrictions on the functions ϕ_i, the IFS contraction factors c_i and the measure μ, T is a contraction map from $\mathcal{L}^p(X, \mu)$ into itself. The completeness of $\mathcal{L}^p(X, \mu)$ guarantees the existence of a unique fixed point $\overline{u}$ of T in $\mathcal{L}^p(X, \mu)$.

2. THE INVERSE PROBLEM

The problem of representing a given image (or a function) by either IFS, IFSP (and variations), IFZS or IFSM is a typical *inverse* problem. Such an inverse problem is, in turn, related to the problem of finding the image/function as the fixed element of a given iteration algorithm of the type IFS, IFSP, IFZS or IFSM on $\mathcal{L}^p$. As we have shown, the problem reduces to the mathematical problem of finding:

1. a suitable metric space Y in which to represent the image (function),

2. a suitable metric d_Y on Y,

3. a suitable contraction map $T : Y \to Y$.

The fact that this main problem has more than one solution leaves room for the search of different kinds of optimality.

We have already provided a solution to the inverse problem for measure approximation using IFSP.[14] In this paper, we outline a solution to the inverse problem for function/image approximation using IFSM. In an accompanying paper,[15] to be referred to as Paper II, we outline an algorithm for the construction of IFSM approximations of arbitrary accuracy to functions and images along with some numerical computations. Because of space limitations, theorems are presented below without proofs. A detailed discussion, complete with proofs will appear elsewhere.[13]

Throughout this paper, we shall employ the following additional notation:

- $Con(X)$: as above, the set of contraction maps on X. We denote the contractivity factor of $w \in Con(X)$ as

$$c := \sup_{x,y \in X, x \neq y} d(w(x), w(y))/d(x,y). \tag{15}$$

  In applications, we shall be primarily concerned with contractive *similitude* maps on X, i.e. inversions, rotations, reflections followed by translations. For convenience, we denote this set of maps as $Sim(X) \subset Con(X)$, i.e.

- $Sim(X) = \{w : X \to X \mid d(w(x), w(y)) = cd(x,y) \text{ for some } c \in [0,1), \forall x, y \in X\}$.

- $d_{Con(X)}$: a metric on the function space $Con(X)$. For $f, g \in Con(X)$,

$$d_{Con(X)}(f,g) = \sup_{x \in X} d(f(x), g(x)). \tag{16}$$

- $\mathcal{M}(X)$: as before, the set of measures on $\mathcal{B}(X)$, the σ-algebra of Borel subsets of X. In the special case that $X \subset \mathbf{R}^D$, let $m^{(D)} \in \mathcal{M}(X)$ denote the Lebesgue measure on $\mathcal{B}(X)$.

- $I_A(x)$: the indicator function of a set $A \subseteq X$. $I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if otherwise.} \end{cases}$

- $Lip(\mathbf{R}) = \{\phi : \mathbf{R} \to \mathbf{R} \mid |\phi(t_1) - \phi(t_2)| \leq K|t_1 - t_2|, \forall t_1, t_2 \in \mathbf{R} \text{ for some } K \in [0, \infty)\}$.

IFS methods are generally based upon Banach's Fixed Point Theorem or Contraction Mapping Principle (CMP) as well as two simple yet powerful consequences. For convenience, we state these results below.

Theorem 1 *(CMP) Let (Y, d_Y) be a complete metric space. Suppose there exists a mapping $f \in Con(Y)$ with contractivity factor $c \in [0,1)$. Then there exists a unique $\overline{y} \in Y$ such that $f(\overline{y}) = \overline{y}$. Moreover, for any $y \in Y$, $d_Y(f^n(y), \overline{y}) \to 0$ as $n \to \infty$.*

The following result has often been referred to in the IFS literature as the "Collage Theorem"[16,1] .

Theorem 2 *Let (Y, d_Y) be a complete metric space. Given a $y \in Y$ suppose that there exists a map $f \in Con(Y)$ with contractivity factor $c \in [0,1)$ such that $d_Y(y, f(y)) < \epsilon$. If $\overline{y}$ is the fixed point of f, i.e. $f(\overline{y}) = \overline{y}$, then $d_Y(y, \overline{y}) < \epsilon/(1-c)$.*

Finally, the following establishes the continuity of fixed points of contraction maps on (Y, d_Y).

Theorem 3 *Let (Y, d_Y) be a metric space and $f, g \in Con(Y)$ with fixed points $\overline{y}_f$ and $\overline{y}_g$, respectively. Then*

$$d_Y(\overline{y}_f, \overline{y}_g) < \frac{1}{1 - c_f} d_{Con(Y)}(f, g), \tag{17}$$

where c_f is the contractivity factor of f.

This result was used to derive continuity properties of IFS attractors and IFSP invariant measures[17] as well as attractors of IFZS[18] and IFSM[13] .

3. IFS WITH MAPS (IFSM) ON $\mathcal{L}^p(X,\mu)$

Having defined above an N-map IFSM $(\mathbf{w},\Phi)$ with associated operator T, we now establish some sufficient conditions on the IFS and grey level maps to ensure that T maps $\mathcal{L}^p(X,\mu)$ into itself.

Proposition 1 *Let $(\mathbf{w},\Phi)$ denote an N-map IFSM with associated operator T defined above. Assume that:*

1. *For any $u\in\mathcal{L}^p(X,\mu)$, $u\circ w_k^{-1}\in\mathcal{L}^p(X,\mu)$, $1\le k\le N$, and*
2. *$\phi_k\in Lip(\mathbf{R})$, $1\le k\le N$.*

Then for $1\le p<\infty$, $T:\mathcal{L}^p(X,\mu)\to\mathcal{L}^p(X,\mu)$.

Proposition 2 *Let $X\subset\mathbf{R}^D$, $D\in\{1,2,...\}$, and $\mu=m^{(D)}$. Let $(\mathbf{w},\Phi)$ be an N-map IFSM such that $w_k\in Sim(X)$ and $\phi_k\in Lip(\mathbf{R})$ for $1\le k\le N$. Then for a $p\in[1,\infty)$ and $u,v\in\mathcal{L}^p(X,\mu)$,*

$$d_p(Tu,Tv)\le C(D,p)d_p(u,v), \tag{18}$$

where

$$C(D,p)=\sum_{k=1}^{N}c_k^{D/p}K_k^p. \tag{19}$$

Proof: For $u,v\in\mathcal{L}^p(X,\mu)$,

$$\begin{aligned}
\| Tu-Tv\|_p &= [\int_X|\sum_{k=1}^{N}{}'[\phi_k(u(w_k^{-1}(x)))-\phi_k(v(w_k^{-1}(x)))]|^p dx]^{1/p}\\
&\le \sum_{k=1}^{N}[\int_{X_k}|\phi_k(u(w_k^{-1}(x)))-\phi_k(v(w_k^{-1}(x)))|^p dx]^{1/p}\\
&= \sum_{k=1}^{N}c_k^{D/p}[\int_X|\phi_k(u(y))-\phi_k(v(y))|^p dy]^{1/p}\\
&\le \sum_{k=1}^{N}c_k^{D/p}K_k[\int_X|u(y)-v(y)|^p dy]^{1/p}\\
&= [\sum_{k=1}^{N}c_k^{D/p}K_k]\,\| u-v\|_p\,.
\end{aligned} \tag{20}$$

Remark 1 *If $C(D,p)<1$, then T is contractive over the space $(\mathcal{L}^p(X,m^{(D)}),d_p)$ and possesses a unique fixed point $\overline{u}$.*

Remark 2 *If $0\le c_k\le 1$ for $1\le k\le N$ then $C(D,p)<1$ for some $p>1$ implies that $C(D,q)<1$ for all $1\le q<p$.*

Remark 3 *Note that for T to be contractive, i.e. $C(D,p)<1$, we can relax the restriction that all IFS maps be contractive, i.e. that $c_k<1$ for $1\le k\le N$.*

Remark 4 *In the special case that the sets $w_k(X)$ do not overlap, a stronger estimate for the Lipschitz constant $C(D,p)$ can be obtained, namely,*

$$C(D,p) = [\sum_{k=1}^{N} c_k^D K_k^p]^{1/p}. \tag{21}$$

Example 1 $X = [0,1]$, $\mu = m^{(1)}$, $N = 3$, $w_i(x) = \frac{1}{3}(x + i - 1), i = 1,2,3$, *with grey level maps* $\phi_1(t) = \frac{1}{2}t$, $\phi_2(t) = \frac{1}{2}$, $\phi_3(t) = \frac{1}{2}t + \frac{1}{2}$. *The fixed point $\overline{u}(x)$ is (up to an equivalence class) the "Devil's staircase function": $\overline{u}(x)$ is continuous at all $x \in X$ and differentiable for all $X \setminus C$, where C denotes the ternary Cantor set on [0,1].*

Example 2 $X = [0,1]$, $N = 3$, $\mu = m^{(1)}$, $w_i(x) = \frac{1}{3}(x + i - 1), i = 1,2,3$, *with grey level maps* $\phi_1(t) = \frac{1}{4}t$, $\phi_2(t) = \frac{1}{4}t$, $\phi_3(t) = 2t$. *Then $\overline{u} := 0$ is a fixed point of T. However, T is contractive only on $\mathcal{L}^1(X, m^{(1)})$.*

3.1 Affine IFSM on $\mathcal{L}^p(X,\mu)$

In applications, it is convenient to employ affine IFS maps $w_k \in Sim(X)$ as well as affine grey level maps ϕ_i. The latter have the form

$$\phi_k(t) = \alpha_k t + \beta_k, \quad t \in \mathbf{R}, \quad k = 1, 2, ..., N. \tag{22}$$

We shall refer to such a system $(\mathbf{w}, \Phi)$ as an *affine IFSM*. The action of the operator T associated with an affine IFSM may be written as follows: For $u \in \mathcal{L}^p(X,\mu)$,

$$(Tu)(x) = \sum_{k=1}^{N} [\alpha_k u(w_k^{-1}(x)) + \beta_k I_{w_k(X)}(x)]. \tag{23}$$

From Proposition 2, if $X \subset \mathbf{R}^D$, then for $u, v \in \mathcal{L}^p(X, m^{(D)})$,

$$d_p(Tu, Tv) \leq C(D,p) d_p(u,v), \tag{24}$$

where

$$C(D,p) = [\sum_{i=1}^{N} c_i^{D/p} |\alpha_i|]. \tag{25}$$

Remark 5 *If $\beta_k = 0$ for $1 \leq k \leq N$, then $\overline{u}(x) := 0$ is a fixed point of T.*

Remark 6 *Let $X = [0,1]$ and $\mu = m^{(1)}$ with $w_i(x) = s_i x + a_i$ and $c_i = |s_i| < 1, 1 \leq i \leq N$. If T is contractive with fixed point $\overline{u}$, then*

$$\begin{aligned} \overline{u}(x) &= \sum_{k=1}^{N}{}' \alpha_k \overline{u}(\frac{x - a_k}{s_k}) + \beta_k I_{w_k(X)}(x) & (26) \\ &= \sum_{k=1}^{N}{}' [\alpha_k \psi_k(x) + \beta_k \chi_k(x)]. & (27) \end{aligned}$$

In other words, $\overline{u}$ may be written as a linear combination of both piecewise constant functions $\chi_k(x)$ as well as functions $\psi_k(x)$ which are obtained by dilatations and translations of $\overline{u}(x)$ and $I_X(x) = 1$, respectively. This is reminiscent of the role of scaling functions in wavelet theory.

Theorem 4 *Let $X \subset \mathbf{R}^D$ and $\mu \in \mathcal{M}(X)$. Define $\mathcal{L}^p_A(X,\mu) \subset \mathcal{L}^p(X,\mu)$ to be the set of fixed points $\overline{u}$ of all contractive N-map affine IFSM $(\mathbf{w},\Phi)$ for $1 \le N < \infty$, on X. Then $\mathcal{L}^p_A(X,\mu)$ is dense in $(\mathcal{L}^p(X,\mu), d_p)$.*

The above result is a consequence of the fact[19] that the set $\mathcal{S}(X)$ of step functions in X is dense in $\mathcal{L}^p(X,\mu)$.

3.2 Infinite-Dimensional Affine IFSM on $\mathcal{L}^p(X,\mu)$

It seems intuitively obvious that the accuracy to which we can approximate a function by the attractor of an IFSM will increase with the number of maps N used in the IFSM. For this reason, we now consider IFSM with an infinite number of IFS and grey level maps, i.e. $N = \infty$. However, for such systems be useful in the solution to the inverse problem on $\mathcal{L}^p(X,\mu)$, we impose a condition on the IFS maps according to the following definitions.

Definition 1 *Let (X,d) be a compact metric space and $\mu \in \mathcal{M}(X)$. A family $\mathcal{A}$ of subsets $A = \{A_i\}$ of X is "μ-dense" in a family $\mathcal{B}$ of subsets B of X if for every $\epsilon > 0$ and any $B \in \mathcal{B}$ there exists a collection $A \subset \mathcal{A}$ such that $A \subseteq B$ and $\mu(B \setminus A) < \epsilon$.*

Definition 2 *Let $\mathbf{w} = \{w_1, w_2, ...\}$, $w_i \in Con(X)$ be an infinite-dimensional IFS. We say that $\mathbf{w}$ generates a "μ-dense and nonoverlapping" — to be abbreviated as "μ-d-n" — family $\mathcal{A}$ of subsets of X if for every $\epsilon > 0$ and every $B \subseteq X$ there exists a finite set of integers $i_k \ge 1, 1 \le k \le N$, such that*

1. $A := \bigcup_{k=1}^N w_{i_k}(X) \subseteq B$,
2. $\mu(B \setminus A) < \epsilon$ *and*
3. $\mu(w_{i_k}(X) \cap w_{i_l}(X)) = 0$ *if* $k \ne l$.

The μ-d-n property allows us to include, within a single infinite-dimensional IFS, an infinite set of finite-dimensional IFS which provide arbitrarily small degrees of refinement. A useful set of affine maps satisfying such a condition on $X = [0,1]$ with respect to Lebesgue measure $m^{(1)}$ is given by the following "wavelet-type" functions:

$$w_{ij}(x) = 2^{-i}(x + j - 1), \quad i = 1, 2, ..., \quad j = 1, 2, \dots, 2^i. \tag{28}$$

For each $i^* \ge 1$, the set of maps $\{w_{i^*j}, j = 1, 2, \dots, 2^{i^*}\}$ provides a set of 2^{-i^*} μ–nonoverlapping contractions of $[0,1]$ which tile $[0,1]$.

Theorem 5 *Let $X \subset \mathbf{R}^D$, $\mu = m^{(D)}$ and $p \ge 1$. Also let $\mathbf{w} = \{w_1, w_2, ...\}$, $w_i \in Sim(X)$, such that $\mathbf{w}$ generates a μ-d-n family of subsets of X. Now define the sequences,*

$$\begin{aligned} \mathbf{a} &= \{\alpha = (\alpha_1, \alpha_2, ...), \alpha_k \in \mathbf{R}, \quad \sum_{k=1}^{\infty} c_k^D |\alpha_k|^p < 1\}, \\ \mathbf{b} &= \{\beta = (\beta_1, \beta_2, ...), \beta_k \in \mathbf{R}\}. \end{aligned} \tag{29}$$

Then

1. *The operator T associated with the affine IFSM $(\mathbf{w},\Phi)$, where $\phi_i(t) = \alpha_i t + \beta_i, i = 1, 2, ...$, with $\alpha \in \mathbf{a}$ and $\beta \in \mathbf{b}$, is contractive on $(\mathcal{L}^p(X, m^{(D)}), d_p)$.*
2. *Define $\mathcal{S}^p = \{u \in \mathcal{L}^p(X, m^{(D)}), Tu = u$ for some $(\alpha,\beta) \in (\mathbf{a},\mathbf{b})\}$. Then $\mathcal{S}^p$ is dense in $(\mathcal{L}^p(X,m), d_p)$.*

Now suppose that $(\mathbf{w}, \Phi)$ is an infinite-dimensional affine IFSM with contractive T operator on $\mathcal{L}^p(X, \mu)$. Then T possesses a fixed point $\overline{u} \in \mathcal{L}^p(X, \mu)$, i.e. $T\overline{u} = \overline{u}$. From Eq. (22), we may write

$$\overline{u}(x) = \sum_{k=1}^{\infty}(\alpha_k \psi_k(x) + \beta_k \chi_k(x)), \tag{30}$$

where $\psi_k(x) = \overline{u}(w_k^{-1}(x))$, $k = 1, 2, \ldots$. We may regard Eq. (29) in two ways: (i) as a Fourier expansion of $\overline{u}(x)$ in the set of functions (assuming that $\overline{u}$ is not a constant) $\{\psi_k, \chi_k\}_{k=1}^{\infty}$ or (ii) as a mixed "wavelet-type" expansion, where $\overline{u}$ and the constant function $I_X(x) = 1$ are *scaling functions* from which the functions ψ_k and χ_k, respectively, are obtained through the dilations and translations produced by the w_k^{-1}.

4. THE INVERSE PROBLEM FOR IFSM ON $\mathcal{L}^P(X, \mu)$

From the Collage Theorem (Theorem 2), the inverse problem for the approximation of functions in $\mathcal{L}^p(X, \mu)$ by IFSM may now be posed as follows:

Given a target function $v \in \mathcal{L}^p(X, \mu)$ and a $\delta > 0$, find an IFSM $(\mathbf{w}, \Phi)$ with associated operator T such that $d_p(v, Tv) = \| v - Tv \|_p < \delta$.

Both our formal solution to this problem as well as the associated algorithm are based on a strategy used in solving the inverse problem of measure approximation using IFSP.[14] Instead of working with a finite set of IFS maps w_i and associated maps ϕ_i, all of which would have to be optimized, we work with an infinite set of *fixed* IFS maps satisfying the μ-d-n property on X. As such, the w_i are considered to form a fixed basis for the representation of compact subsets of X. Only an optimization over the grey level maps ϕ_i is required.

Now let

$$\mathbf{w}^N = \{w_1, w_2, \ldots, w_N\}, \quad N = 1, 2, \ldots, \tag{31}$$

denote N-map truncations of $\mathbf{w}$. Also let

$$\Phi^N = \{\phi_1, \phi_2, \ldots, \phi_N\}, \tag{32}$$

denote an associated N-vector of grey level maps with the restriction that the $\phi_i \in Lip(\mathbf{R})$. Let $T^N : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$ be the operator associated with the N-map IFSM $(\mathbf{w}^N, \Phi^N)$. Given a target function $v \in \mathcal{L}^p(X, \mu)$, the following result ensures that the collage distance $\| v - T^N v \|_p$ can be made arbitrarily small.

Theorem 6 *Let $v \in \mathcal{L}^p(X, \mu)$, where $p \in [1, \infty)$. Assume that $\mathbf{w} = \{w_1, w_2, \ldots\}$, $w_i \in Con(X)$, such that $\mathbf{w}$ generates a μ-d-n family $\mathcal{A}$ of subsets of X. Define the N-map truncations $(\mathbf{w}^N, \Phi^N)$ as above, with $\phi_i \in Lip(\mathbf{R})$. Then*

$$\liminf_{N \to \infty} \| v - T^N v \|_p = 0. \tag{33}$$

In our following paper (II. Algorithm and computations),[15] we describe an algorithm for the construction of IFSM approximations of arbitrary accuracy to a target set $v \in \mathcal{L}^2(X, \mu)$.

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