# Generalized Self-Similarity, Wavelets and Image Analysis

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**Abstract**

We present a solution to a functional equation by means of the construction of a contractive operator on some functional space. This solutions presents a kind of self-similarity and enables us to generalize the model introduced by Cabrelli et al. in [CFMV92] allowing a much greater flexibility. In particular dilation equations of the type $f(x) = \sum c_k f(2x-k)$ fit into this model, and hence we can construct a multiresolution analysis in the sense of Mallat and Meyer.
On the other hand, this "generalized self-similarity" notion provides us with a method for the construction of an operator whose fixed point is close to a given target.

*Keywords:* Self-Similarity, Functional Equation, Dilation Equation, Refinement Equation, Wavelets, Fixed Points, Fractals.

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# 1 Introduction

Self-Similar objects are those that can be constructed out of smaller copies of itself. When we deal with sets, this concept can be formulated using the notion of Iterated Function Schemes (IFS) ([Hut81], [Bar88]): If $(X, d)$ is a metric space and $\Phi = \{w_1, \ldots, w_n\}$ ($w_i : X \to X$ , $\{w_i\}_{i=1,\ldots,N}$) is a set of maps, then $\mathcal{A} \subset X$ is self-similar with respect to $\Phi$ if $\mathcal{A} = \cup w_i(\mathcal{A})$. It can be shown, that if $X$ is complete, and the maps are contractive, then there exists a unique compact self-similar set with respect to $\Phi$.

This concept can be extended in different ways to different kind of objects: self-similar measures can also be defined using IFS (see [Hut81], [Bar88]) and recently have been studied by Strichartz using Fourier and Wavelet analysis ([Str91], [Str94]).

Aiming to recover self-similarity parameters of physical signals, Hwang and Mallat study the self-similarity of the wavelet transform ([HM94]).

One way to extend the notion of self-similarity to functions, is to require that the graph of the function should be a self-similar set.
If the function is defined on a self-similar set, then we could require that the function share the self-similarity of the domain, i.e.: if $X = \cup_{i=1}^{n} w_i(X)$, then

$$f(w_i(x)) = f(x), \quad i = 1, \ldots, n. \tag{1.1}$$

For this definition we require the $w_i$ to be disjoint (i.e. $w_i(X) \cap w_j(X) = \emptyset$, $i \neq j$ ).

From IFS-theory it can be shown that if $f$ is a continuos function satisfying the self-similarity condition (1.1), $f$ has to be constant.

In order to consider more general solutions, we relax the condition of self-similarity (1.1), introducing a set of functions $\varphi_1, \ldots, \varphi_n$ and requiring that $f$ satisfy

$$\varphi_i(f(w_i^{-1}(x))) = f(x) \quad x \in w_i(X) \quad i = 1, \ldots, n. \tag{1.2}$$

Finally, to allow overlapping maps in the IFS, we introduce a function $\mathcal{O}$ that combines the values of $\varphi_i \circ f \circ w_i^{-1}(x)$ for $i = 1, \ldots, n$ for the same $x$.

In this paper we will study the existence of self-similar functions in different contexts and relax even more the self-similarity condition (1.2) allowing *space-dependent* $\varphi_i$'s and $\mathcal{O}$.

The problem of finding a function $\mathbf{u}$ that satisfies a self-similarity equation of the type:

$$\mathbf{u}(x) = \mathcal{O}\,(x, (\mathbf{u} \circ g_1)(x), \dots, (\mathbf{u} \circ g_r)(x)), \tag{1.3}$$

has been studied by Bajraktarevic in 1957 ([Baj57]). In the same year, a similar equation was considered by de Rahm ([dR57]), and conditions for continuous solutions were found.

In [Hut81] Hutchinson, extending the concept of self-similarity to parametric curves, considered a particular case of this equation.

More recently, related functional equations were studied in fractal interpolation, in order to show the existence and construction of continuous fractal functions ([Bar86], [BH89], [Dub85], [Dub86], [DE90], [Hat85], [Hat86]) .

In this paper we study a slightly more general equation

$$\mathbf{u}(x) = \mathcal{O}(x, \varphi_1(x, (\mathbf{u} \circ g_1)(x)), \dots, \varphi_r(x, (\mathbf{u} \circ g_r)(x))\,). \tag{1.4}$$

that encloses most of the cases mentioned before and generalizes the concept of self-similar function (1.1). We find conditions on the components in order to assure the existence of solutions.

We construct an operator on a suitable function space and the solution of our equation will be a fixed point of this operator. This not only yields a solution of the equation, but also shows that this solution can be computationally efficiently calculated: we obtain it by iterating the operator.

Cabrelli et al. in [CFMV92] constructed an operator of the type (1.4), but neither $\mathcal{O}$ nor the functions $\varphi_i$ were allowed to depend on $x$. In addition, the $\varphi_i$'s had to satisfy some pretty restrictive conditions.
In the present approach, we were able to remove most of these conditions making this model much more versatile and therefore more suitable for applications. For example we can use it to model two situations which are of general interest: the construction of wavelets and image or signal analysis.

In the first case, one wants to find solutions to a *dilation equation* of the type

$$\phi(x) = \sum_{k \in Z} c_k \, \phi(2x - k), \tag{1.5}$$

in order to then construct a wavelet basis in $\mathcal{L}^2(\mathcal{R})$. Suitably defining $\varphi_i$, $g_i$ and $\mathcal{O}$ in (1.4) will yield (1.5) We will expand on this idea later in the paper. Naturally these functions will depend on the coefficients $c_k$, hence the existence of a solution will be translated into a condition on the coefficients $c_k$. Considering the attention that has been given to this equation lately, we think that it is interesting to observe how natural it fits into this model, and how the conditions on the coefficients appear in a very simple manner.

In addition, the higher dimensional parallel of (1.5) can be addressed in the same way; in the particular case that all the $c_k$ are equal, we obtain the equation studied by Gröchenig and Madych in [GM92] and Strichartz in [Str93]. Currently there is a growing interest in "multiwavelets", which can be constructed using dilation equations in which the coefficients are matrices and the solutions vector-valued functions ([GHM94], [HC94]). Solutions to these matrix dilation equations using the concept of generalized self-similar functions are studied in a joined work with C. Heil ([CCM94]).

In the second case, in signal processing, in particular in image representation, a well known problem is the design of an adaptive code for a given target. This has been studied in particular using fractals and self-similar models (see [AT89], [BEHL85], [BS88], [CFMV92], [CM91], [DGS94], [HM90]). Some of the advantages of this approach are the compression rates achieved, and the complexity of the images that can be represented. Our strategy consists in finding an operator $T$, whose fixed point is the given target. Together with a least-square-approximation argument we are able to propose a basic method for the construction of such an operator. The functional equation considered, represents a generalization of the concept of selfsimilar function extending the applicability of the model to a wider class of images and allowing more flexibility in the choice of the parameters.

The outline of the paper is as follows: we first study in section 2 conditions for the existence of solutions to the functional equation. In section 3 we will show the application to the dilation equation in $\mathcal{L}^1(\mathcal{R})$ and finally, in section 4 we outline the construction of an operator whose fixed point is a predetermined target.

# 2 The self-similarity equation

We will consider the functional equation (1.4)on two different settings:

## 2.1 B(X,E)-case

Let $(X,d)$ be a compact metric space and $(E,\ell)$ a metric space where $E$ is a closed subset of $\mathcal{R}^m$ (in particular ($E$ could be $\mathcal{R}^m$) and $\ell$ a distance in $E$ induced by some norm of $\mathcal{R}^m$. Let us also consider a point $t_0 \in E$ that will remain fixed throughout the whole section.

We consider the functional space

$$\mathcal{B}(X,E) = \{\mathbf{u} : X \longrightarrow E,\ \mathbf{u} \text{ bounded}\},$$

with

$$(2.1) \qquad D(\mathbf{u},\mathbf{v}) = \sup_{x\in X} \ \ell\,(\mathbf{u}(x),\mathbf{v}(x)), \quad \forall \mathbf{u},\mathbf{v}, \in \ \mathcal{B}(X,E).$$

It is well-known that $(\mathcal{B}(X,E),D)$ is a complete metric space.

Let us now define the functions $\mathcal{O}$, $w_i, \varphi_i, i = 1,\ldots,r$ in order to construct an operator $\mathcal{T}$ on $\mathcal{B}(X,E)$.

Let $\mathcal{O} : X \times E^r \longrightarrow E$ be non-expansive for each $x \in X$ , i.e. :

$$(2.2) \qquad \ell\,(\mathcal{O}\,(x,\overrightarrow{k^1}),\mathcal{O}\,(x,\overrightarrow{k^2})) \leq \sup_{1\leq i\leq r} \ell\,(k_i^1,k_i^2) \ \ \forall \overrightarrow{k^1},\overrightarrow{k^2} \ \in E^r.$$

Let $w_i : X \to X,\ i = 1,\ldots,r$ be $r$ injective maps, which are not necessarily contractive, and let $\varphi_i : X\times E \to E,\ i = 1,\ldots r$ be $r$ functions that for each $x \in X$ satisfy the Lipschitz condition:

$$(2.3) \quad \ell\,(\varphi_i(x,k_1),\varphi_i(x,k_2)) \leq c\,\ell\,(k_1,k_2), \ \ \forall k_1,k_2 \in E, \ \ i = 1,\ldots,r$$

where $c \geq 0$ does not depend on $x$.

In order to be able to define an operator on $\mathcal{B}(X, E)$ , we need some stability conditions. We define a function $f$ to be *stable*, if $f(A)$ is bounded, whenever $A$ is a bounded set. Hence we shall assume that $\mathcal{O}$ and $\varphi_i, i = 1, \ldots, r$ are stable.

Now we define an operator $\mathcal{T}$ on $\mathcal{B}(X, E)$ in the following way:

$$(\mathcal{T}\mathbf{u})(x) = \mathcal{O}\,(x, \varphi_1(x, \tilde{\mathbf{u}}_1(x)), \ldots, \varphi_r(x, \tilde{\mathbf{u}}_r(x)); \tag{2.4}$$

$$\text{where} \quad \tilde{\mathbf{u}}_i(x) = \begin{cases} \mathbf{u}(w_i^{-1}(x) & \text{if } x \in \mathrm{Img}(w_i) \\ t_0 & \text{otherwise} \end{cases} \qquad 1 \leq i \leq r. \tag{2.5}$$

We shall use $\mathcal{O}\,(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{u}}_i(x))})$ for the right hand side of (2.4). We can prove the following

**Theorem 2.1.1** *With the above notation, if $c > 0$ is the Lipschitz constant for the $\varphi_i$'s, then*

$$\mathcal{T} : \mathcal{B}(X, E) \longrightarrow \mathcal{B}(X, E), \qquad \textit{and}$$

$$D(\mathcal{T}\boldsymbol{u}, \mathcal{T}\boldsymbol{v}) \leq c\, D(\boldsymbol{u}, \boldsymbol{v}).$$

*In particular, if $c < 1$, $\mathcal{T}$ is contractive and therefore there exists a unique $\boldsymbol{u}^*$ in $\mathcal{B}(X, E)$ such that $\mathcal{T}\boldsymbol{u}^* = \boldsymbol{u}^*$.*

**Proof :** If $\mathbf{u} \in \mathcal{B}(X, E)$ then it is easy to verify that $\mathcal{T}\mathbf{u} \in \mathcal{B}(X, E)$. Now if $\mathbf{u}, \mathbf{v} \in \mathcal{B}(X, E)$ then

$$\begin{aligned}
\ell\,((\mathcal{T}\mathbf{u})(x), (\mathcal{T}\mathbf{v})(x)) &= \ell\,(\,\mathcal{O}\,(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{u}}_i(x))}), \mathcal{O}\,(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{v}}_i(x))})\,) \\
&\leq \sup_{1 \leq i \leq r} \ell\,(\,\varphi_i(x, \tilde{\mathbf{u}}_i(x)), \varphi_i(x, \tilde{\mathbf{v}}_i(x))\,) \\
&\leq \sup_{1 \leq i \leq r} c\;\ell\,(\,\tilde{\mathbf{u}}_i(x), \tilde{\mathbf{v}}_i(x)\,) \\
&\leq c\; \sup_{y \in X} \ell\,(\,\mathbf{u}(y), \mathbf{v}(y)\,) \\
&= c\; D(\mathbf{u}, \mathbf{v}).
\end{aligned}$$

Therefore

$$D(\mathcal{T}\mathbf{u}, \mathcal{T}\mathbf{v}) \leq c\; D(\mathbf{u}, \mathbf{v}).$$

■

We then have the following

**Corollary 2.1.1** *If $c < 1$, the functional equation*

$$\boldsymbol{u} = \mathcal{O}(x, \varphi_1(x, \tilde{\boldsymbol{u}}_1(x)), \ldots, \varphi_r(x, \tilde{\boldsymbol{u}}_r(x))\,) \tag{2.6}$$

*where the $\tilde{\mathbf{u}}_i$ are as in (2.5), has a unique solution in $\mathcal{B}(X, E)$.*

**Proof :**

The fixed point of the operator $\mathcal{T}$ is the solution of the equation. ■

Note that (2.6) is a generalization of the original functional equation given in (1.3).

In what follows, we will study the operator (2.4) in the $\mathcal{L}^{\mathbf{p}}$spaces.

## 2.2 $L^P$-case

Let now $X \subseteq \mathcal{R}^n$ compact, with $\mu$ the $n$-dimensional Lebesgue measure and let $E = \mathcal{R}^m$ with some norm $||.||$. (Note: $E$ could be chosen to be any Banach space.) If we consider the functions $\mathbf{u} : X \to E$ such that the real-valued function $||\mathbf{u}(.)||$ is Lebesgue-measurable, and, as usual, functions that are equal almost everywhere are identified.

If $1 \leq p < +\infty$, let

$$\mathcal{L}^{\mathbf{p}}(X, E) = \{\mathbf{u} : X \to E : \int_X ||\mathbf{u}(x)||^p d\mu(x) < +\infty\}$$

with $||\mathbf{u}||_p = \left(\int_X ||\mathbf{u}(x)||^p d\mu(x)\right)^{1/p}$; and

$$\mathcal{L}^{\infty}(X, E) = \{\mathbf{u} : X \to E : \ ||\mathbf{u}(.)|| \quad \text{essentially bounded}\}$$

with $||\mathbf{u}||_\infty = \text{ess.sup.}||\mathbf{u}(.)||$.

It is well known, that $\mathcal{L}^{\mathbf{p}}(X, E)$, $1 \leq p \leq +\infty$ is a Banach space.

For measurable $\mathbf{u} : X \to E$ we define as before the operator (2.4),

$$(\mathcal{T}\mathbf{u})(x) = \mathcal{O}\,(x, \varphi_1(x, \tilde{\mathbf{u}}_1(x)), \ldots, \varphi_r(x, \tilde{\mathbf{u}}_r(x)),$$

where the $w_i's, \varphi_i's$ and $\mathcal{O}$ are as in the previous section, with the following additional conditions:

1. The maps $\{w_i\}$ satisfy a Lipschitz condition, i.e. there exists $s > 0$, such that $d(w_i(x), w_i(y)) \leq s\, d(x, y)$ where $d$ is the Euclidean distance in $\mathcal{R}^n$.
2. The functions $\varphi_i,\ i = 1, \ldots, r$ and $\mathcal{O}$ are Borel measurable.

These additional conditions are required in order to guarantee the measurability of $\mathcal{T}\mathbf{u}$.

We have the following

**Proposition 2.2.1** *Let $\mathcal{T}$ be defined as above, then $\mathcal{T}\boldsymbol{u} : X \to E$ is measurable for each measurable function $\boldsymbol{u} : X \to E$ and also if $\boldsymbol{u}, \boldsymbol{v}$ are measurable and $\boldsymbol{u} = \boldsymbol{v}$ a.e. then $\mathcal{T}\boldsymbol{u} = \mathcal{T}\boldsymbol{v}$ a.e.*

**Proof:** The measurability of $\mathcal{T}\mathbf{u}$ for measurable $\mathbf{u}$ is a consequence of the stability and the Borel-measurability of $\mathcal{O}$ and the $\varphi_i$'s and the fact that the $w_i$'s are Lipschitz. Now if $Z = \{x :\ \mathbf{u}(x) \neq \mathbf{v}(x)\}$ then $\{x : \mathcal{T}\mathbf{u}(x) \neq \mathcal{T}\mathbf{v}(x)\} \subset \cup_{i=1}^{r} w_i(Z)$. The Lipschitz condition of the $w_i$'s implies that $\mu(w_i(Z)) = 0$ if $\mu(Z) = 0$ and therefore the result follows. ■

Now we consider first the space $\mathcal{L}^\infty$ defined before. The case $\mathcal{L}^\mathbf{p}$ $1 \leq p < +\infty$ will be treated later.

**Theorem 2.2.1** *Let $\mathcal{T}$ be the operator of proposition 2.2.1. Then, $\mathcal{T} : \mathcal{L}^\infty \to \mathcal{L}^\infty$ and*

$$||\mathcal{T}\boldsymbol{u} - \mathcal{T}\boldsymbol{v}||_\infty \leq c||\boldsymbol{u} - \boldsymbol{v}||_\infty, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{L}^\infty.$$

**Proof:** If $\mathbf{u} \in \mathcal{L}^\infty$ then let $Z \subset X$, $\mu(Z) = 0$ and $\mathbf{u}$ bounded in $X - Z$. If we define $\mathbf{v} : X \to E$ by $\mathbf{v} = \mathbf{u}\,\mathcal{X}_{X-Z}$, then $\mathbf{v} = \mathbf{u}$ a.e. and $\mathbf{v}$ is bounded. Then $\mathcal{T}\mathbf{v}$ is bounded and using the preceding proposition, $\mathcal{T}\mathbf{u} = \mathcal{T}\mathbf{v}$ a.e. and therefore $\mathcal{T}\mathbf{u} \in \mathcal{L}^\infty$.

From the proof of Theorem 2.1.1 we see that for $\mathbf{u}$ and $\mathbf{v} \in \mathcal{L}^\infty$ we have

$$||(\mathcal{T}\mathbf{u})(x) - (\mathcal{T}\mathbf{v})(x)|| \leq c||\mathbf{u} - \mathbf{v}||_\infty \text{ a.e. on } X,$$

which implies that

$$||\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}||_\infty \leq c\,||\mathbf{u} - \mathbf{v}||_\infty. \qquad ■$$

We will now analyze the case $\mathcal{L}^{\mathbf{p}}$ $1 \le p < \infty$. We have the following

**Theorem 2.2.2** *Let $\mathcal{T}$ be the operator of proposition 2.2.1. Then, if $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{L}^p(X,E)$, then $(\mathcal{T}\boldsymbol{u} - \mathcal{T}\boldsymbol{v}) \in \mathcal{L}^p(X,E)$ and*

$$\|\mathcal{T}\boldsymbol{u} - \mathcal{T}\boldsymbol{v}\|_p \le (r\,s)^{1/p} c \|\boldsymbol{u} - \boldsymbol{v}\|_p \ ,$$

*where $s$ and $c$ are the Lipschitz constants of $w_i$ and $\varphi_i$ respectively. Furthermore the finiteness of $\mu(X)$ yields*

$$\mathcal{T} : \mathcal{L}^p(X,E) \to \mathcal{L}^p(X,E).$$

**Proof :** If $\mathbf{u}, \mathbf{v} \in \mathcal{L}^{\mathbf{p}}$, then by Proposition 2.2.1, $\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}$ is measurable and

$$\begin{aligned}
\|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\|_p^p &= \int_X \|(\mathcal{T}\mathbf{u})(x) - (\mathcal{T}\mathbf{v})(x)\|^p \, d\mu(x) \\
&= \int_X \|\mathcal{O}(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{u}}_i(x))}\,) - \mathcal{O}(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{v}}_i(x))}\,)\,\|^p d\mu(x) \\
&\le \int_X \sup_{1 \le i \le r} \|\, \varphi_i(x, \tilde{\mathbf{u}}_i(x)) - \varphi_i(x, \tilde{\mathbf{v}}_i(x))\,\|^p \, d\mu(x) \quad (\text{ by } 2.2\ ) \\
&\le c^p \int_X \sup_{1 \le i \le r} \|\, \tilde{\mathbf{u}}_i(x) - \tilde{\mathbf{v}}_i(x)\,\|^p \, d\mu(x) \quad (\text{ by } 2.3\ ) \\
&\le c^p \sum_{1 \le i \le r} \int_{w_i(X)} \|\, \mathbf{u}(w_i^{-1}(x)) - \mathbf{v}(w_i^{-1}(x))\,\|^p \, d\mu(x) \\
&\le s\,c^p \sum_{1 \le i \le r} \int_X \|\, \mathbf{u}(t) - \mathbf{v}(t)\,\|^p \, d\mu(t) \\
&= s\,r\,c^p\,(D_p(\mathbf{u}, \mathbf{v}))^p.
\end{aligned}$$

From this inequality we see that if $\mathbf{u}, \mathbf{v} \in \mathcal{L}^{\mathbf{p}}$, then

$$\|\mathcal{T}\mathbf{v}\|_p \le \|\mathcal{T}\mathbf{v} - \mathcal{T}\mathbf{u}\|_p + \|\mathcal{T}\mathbf{u}\|_p \le (sr)^{1/p} c \|\mathbf{u} - \mathbf{v}\|_p + \|\mathcal{T}\mathbf{u}\|_p;$$

what says that if there exists a function $\mathbf{u} \in \mathcal{L}^{\mathbf{p}}$ such that $\mathcal{T}\mathbf{u} \in \mathcal{L}^{\mathbf{p}}$ then $\mathcal{T}$ sends $\mathcal{L}^{\mathbf{p}}$ into $\mathcal{L}^{\mathbf{p}}$, $1 \le p < +\infty$. Now, since $\mu(X) < +\infty$ then $\mathcal{L}^{\infty} \subset \mathcal{L}^{\mathbf{p}}, 1 \le p$ and since, by Theorem 2.2.1 $\mathcal{T} : \mathcal{L}^{\infty} \to \mathcal{L}^{\infty}$, we get the desired result. ∎

**Corollary 2.2.1** *If, with the above notation, $(s\,r)^{1/p}c < 1$ we have that $\mathcal{T}$ is a contraction map on $\mathcal{L}^p$ $1 \leq p \leq \infty$ and the functional equation given by 2.6:*

$$u = \mathcal{O}(x, \varphi_1(x, \tilde{u}_1(x)), \ldots, \varphi_r(x, \tilde{u}_r(x))\,),$$

*has a unique solution in $\mathcal{L}^p$.*

Note that the solution to the functional equation 2.6 presented here can be obtained as the limit of the iteration of the operator $\mathcal{T}$ at any starting function.

# 3 Dilation equations and wavelets

Families of functions

$$\Psi_{ab}(x) = |a|^{-\frac{1}{2}}\Psi(\frac{x-b}{a}) \quad a, b \in \mathcal{R} \tag{3.1}$$

obtained through translations and dilation of a single function $\Psi$ are called *wavelets* (following Grossman and Morlet [GM84]). If in 3.1 we restrict the choice of $a, b$ to a discrete lattice, we obtain

$$\Psi_{jk}(x) = |a_0|^{-\frac{j}{2}}\Psi(a_0^{-j}x - kb_0) \quad j, k \in \mathcal{Z}. \tag{3.2}$$

Here we fixed a dilation step $a_0 > 1$ and a translation step $b_0 \neq 0$. For particular selections of $\Psi, a_0$ and $b_0$ (usually $a_0 = 2, b_0 = 1$), $\Psi_{jk}$ constitute an orthonormal basis. The classical example is the Haar basis, where

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ -1 & \text{if } 1/2 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad a_0 = 2, b_0 = 1.$$

$\Psi_{jk}(x) = 2^{-j/2}\Psi(2^{-j}x - k) \quad j, k \in \mathcal{Z}$ form an unconditional basis for $\mathcal{L}^2(\mathcal{R})$ and also an unconditional basis for all $\mathcal{L}^p(\mathcal{R}), 1 < p < \infty$. There are many other examples of orthonormal bases (see for example Daubechies [Dau88] and Meyer [Mey88] ).

In the case that one restricts the choice of $a_0$ to 2, Mallat and Meyer showed that wavelet basis constructions can be realized by a 'multiresolution analysis'. A multiresolution analysis for $\mathcal{L}^2(\mathcal{R})$ is defined as a sequence $\{\mathcal{V}_n\}_{n\in Z}$ of subspaces of $\mathcal{L}^2(\mathcal{R})$ and a function $\phi$; such that

- 1. $\mathcal{V}_n \subset \mathcal{V}_{n+1}, n \in \mathcal{Z}$
  2. $\bigcap_n \mathcal{V}_n = \emptyset$
  3. $\overline{\bigcup_n \mathcal{V}_n} = \mathcal{L}^2(\mathcal{R})$
  4. $f(t) \in \mathcal{V}_n \Longleftrightarrow f(2t) \in \mathcal{V}_{n+1}$;

- $\phi \in \mathcal{V}_0$ is such that $\{\phi(t-k)\}_{k\in Z}$ form an orthonormal basis for $\mathcal{V}_0$.

Note that, since $\phi \in \mathcal{V}_0 \subset \mathcal{V}_1$, and $\{\sqrt{2}\phi(2t-k)\}_{k\in Z}$ form an orthonormal basis of $\mathcal{V}_1$, then there *must* exist a unique set of coefficients $\{c_k\}_{k\in Z}$, such that

$$\phi(x) = \sum_{k\in Z} c_k \, \phi(2x-k). \tag{3.3}$$

This equation (for arbitrary $\{c_k\}$) is called a *dilation equation*, and $\phi$ is called the scaling function. On the other hand, not every solution of (3.3) generates a multiresolution analysis. It is of great interest to establish conditions on the coefficients $c_k$ in order to obtain smooth solutions that generate multiresolution analysis (see for example [Dau88], [DL91], [DL92], [Hei91], [CH94]). If the $\{c_k\}_{k\in Z}$ satisfy the desired conditions, then the existence of a solution of 3.3 implies the existence of an orthonormal wavelet basis in $\mathcal{L}^2(\mathcal{R})$, namely, define

$$\Psi(x) = \sum_{k\in Z} (-1)^k c_{k+1} \, \phi(2x+k). \tag{3.4}$$

Then $\Psi_{jk}(x) = 2^{-j/2}\Psi(2^{-j}x - k), j, k \in \mathcal{Z}$ will form an orthonormal basis of $L^2(\mathcal{R})$ (see Mallat [Mal89]). Being able to solve (3.3) will then provide us with methods of generating wavelets.

In the case that one is interested in compactly supported solutions, only finitely many $c_k$ are non-zero. Our aim is to find solutions to the dilation equation defined in (3.3), in precisely that case, i.e:

$$\phi(x) = \sum_{k=0}^{N} c_k \, \phi(2x-k). \tag{3.5}$$

Let us first describe the particular choice of the functional space and the resulting contractive operator, and then relate it to the dilation equation.

Let us remark that equations of the type of (3.3) were studied in other contexts and represent a particular case of refinement equations (see [CDM91]).

### 3.1 Self-Similarity Equation for the dilation equation

We are interested in finding compactly supported solutions, and since every compactly supported solution $\phi$ of (3.5) satisfies $\text{supp}(\phi) \subset [0, N]$, then we can restrict ourselves to functions $f : [0, N] \to \mathcal{R}$. If in addition we are looking for continuous solutions of (3.5), we have to require that $f(0) = f(N) = 0$.

It is convenient to establish the following correspondence: To each $f : [0, N] \to \mathcal{R}$ we can associate a function $\mathbf{u}_f : [0, 1] \to \mathcal{R}^N$,

(3.6)
$$\mathbf{u}_f(x) = (\mathbf{u}_1(x), \ldots, \mathbf{u}_N(x)) \quad \text{where} \quad \mathbf{u}_i(x) = f(x + (i-1)), \quad i = 1, \ldots, N .$$

Note that $\mathbf{u}_i(1) = \mathbf{u}_{i+1}(0), i = 1, \ldots, N-1$. On the other hand, if $\mathbf{u} : [0, 1] \to \mathcal{R}^N$ is a function that satisfies that $\mathbf{u}_i(1) = \mathbf{u}_{i+1}(0), i = 1, \ldots, N-1$, then we can associate to $\mathbf{u}$ a function $f_{\mathbf{u}} : [0, N] \to \mathcal{R}$ in the following way:

$$f_{\mathbf{u}}(x) = \sum_{i=1}^{N} \mathcal{X}_{[\frac{i-1}{N}, \frac{i}{N}]}(x) \mathbf{u}_i(x - (i-1)). \tag{3.7}$$

It is straightforward to see, that if $\phi : [0, N] \to \mathcal{R}$ satisfies the dilation equation (3.5), then $\mathbf{u}_\phi$ satisfies

$$\mathbf{u}_\phi(x) = \begin{cases} A_1 \mathbf{u}_\phi(2x) & 0 \le x \le \frac{1}{2} \\ A_2 \mathbf{u}_\phi(2x-1) & \frac{1}{2} < x \le 1 \end{cases} , \tag{3.8}$$

where $A_1$ and $A_2$ are the $N \times N$ matrices associated to (3.5) defined by

$$\begin{aligned} [A_1]_{ij} &= c_{2i-j-1} & 1 \le i, j \le N \\ [A_2]_{ij} &= c_{2i-j} & 1 \le i, j \le N, \end{aligned} \tag{3.9}$$

where the $c_i's$ are the coefficients of the dilation equation.

Reciprocally, if $\mathbf{u}$ satisfies (3.8) and $\mathbf{u}_i(1) = \mathbf{u}_{i+1}(0), i = 1, \ldots, N-1$ then $\phi_{\mathbf{u}}$ satisfies (3.5) i.e:

$$\phi_{\mathbf{u}}(x) = \sum_{k=0}^{N} c_k \, \phi_{\mathbf{u}}(2x - k).$$

We will now apply the results of section 2 to our problem. Let $X = [0,1]$ and $E = \mathcal{R}^N$ with some norm $\|\ \|$. We consider $r = 2$, and define the functions $w_1, w_2, \varphi_1, \varphi_2$ as follows:

$$w_i : [0,1] \longrightarrow [0,1] \quad i = 1,2$$

$$w_1(x) = \frac{1}{2}x \qquad w_2(x) = \frac{1}{2}x + \frac{1}{2}$$

$$\varphi_i : [0,1] \times \mathcal{R}^N \longrightarrow \mathcal{R}^N \quad i = 1,2$$

$$\varphi_i(x, \vec{t}) = A_i \vec{t} \quad i = 1,2.$$

In addition, we define the following map $\mathcal{O}$:

$$\mathcal{O} : [0,1] \times \mathcal{R}^N \times \mathcal{R}^N \longrightarrow \mathcal{R}^N,$$

$$\mathcal{O}(x, \vec{t_1}, \vec{t_2}) = \begin{cases} \vec{t_1} & 0 \le x \le \frac{1}{2} \\ \vec{t_2} & \frac{1}{2} < x \le 1 \end{cases}.$$

$\mathcal{O}$ can also be written as follows:

$$\mathcal{O}(x, \vec{t_1}, \vec{t_2}) = \mathcal{X}_{[0,\frac{1}{2}]}(x)\, \vec{t_1} + \mathcal{X}_{(\frac{1}{2},1]}(x)\, \vec{t_2}.$$

Clearly $\mathcal{O}$ , $\varphi_1$ and $\varphi_2$ are stable and Borel-measurable, $w_i$ both satisfy Lipschitz conditions with Lipschitz constant $s = \frac{1}{2}$, and since $\mathcal{O}$ satisfies:

$$\|\mathcal{O}(x_0, \vec{t_1}, \vec{t_2}) - \mathcal{O}(x_0, \vec{t_1'}, \vec{t_2'})\| = \begin{cases} \|\vec{t_1} - \vec{t_1'}\| & 0 \le x_0 \le \frac{1}{2} \\ \|\vec{t_2} - \vec{t_2'}\| & \frac{1}{2} < x_0 \le 1 \end{cases},$$

$\mathcal{O}$ is non-expansive.

$\varphi_1, \varphi_2$ satisfy a Lipschitz condition with constant $c = c^*$, where

$$c^* = \max\{\sup_{\|\vec{t}\| \le 1} \|A_1 \vec{t}\|, \sup_{\|\vec{t}\| \le 1} \|A_2 \vec{t}\|\}.$$

We are therefore in the setting of the previous section and considering $\mathcal{T}$ as in (2.4), will yield

$$(\mathcal{T}\mathbf{u})(x) = \mathcal{X}_{[0,\frac{1}{2}]}(x) A_1 \mathbf{u}(w_1^{-1}(x)) + \mathcal{X}_{(\frac{1}{2},1]}(x) A_2 \mathbf{u}(w_2^{-1}(x)).$$

Now we can apply theorems 2.1.1 and 2.2.2 to obtain:

$$\begin{array}{llll} (3.10) & D(\mathcal{T}\mathbf{u},\mathcal{T}\mathbf{v}) & \leq & c^* D(\mathbf{u},\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v} \in \mathcal{B}(X,E) \quad \text{and} \\ (3.11) & \|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\|_p & \leq & c^* \|\mathbf{u}-\mathbf{v}\|_p, \qquad \forall \mathbf{u},\mathbf{v} \in \mathcal{L}^{\mathbf{P}}, \quad 1 \leq p \leq \infty. \end{array}$$

We see that the Lipschitz constant is the same for both cases, since $r\,s = 1$. Hence, if $c^* < 1$ we have that $\mathcal{T}$ is a contractive operator in $\mathcal{L}^{\mathbf{P}}$, $1 \leq p \leq \infty$ as well as in $\mathcal{B}(X,E)$, and has a unique fixed point $\mathbf{u}^*$ in both spaces.

Note that up to this point we did not use any particular choice of $\|\,.\,\|$ in $\mathcal{R}^N$; for our purposes it is enough that there exists one norm with the desired properties.
However, even with that consideration, the requirement that $c^* < 1$ seems to be rather strong and uninteresting for most applications. (Note that in this case $\mathcal{T}$ is linear, and hence if it is contractive the fixed point is the 0 function!).

Moreover, if the coefficients of the dilation equation satisfy the so called "accuracy condition"

$$(3.12) \qquad \sum_k c_{2k} = \sum_k c_{2k+1} = 1,$$

then both matrices $A_1$ and $A_2$ have 1 as an eigenvalue, and hence $c^*$ is always $\geq 1$!

Clearly we want to rule out having the zero function as a fixed point, and hence we will concentrate on the behavior of $\mathcal{T}$ on some adequate subset. We will assume that condition (3.12) holds.

It can be seen, that any solution of equation (3.5) satisfies $\sum_{k=1}^N \phi(x+k) = c$ for some $c$. Following the correspondence (3.6), this means that the associated function $\mathbf{u}_\phi$ satisfies $\sum_{i=1}^N \mathbf{u}_i(x) = c$. If we assume without loss of generality that $c = 1$, we can restrict our attention to functions:

$$\mathbf{u} : [0,1] \to M$$

$$\text{where} \quad M = \{\vec{x} \in \mathcal{R}^N : \sum_{i=1}^N x_i = 1\}$$

Since the columns of $A_1$ and $A_2$ add up to 1, we have that $M$ is invariant under $A_1$ and $A_2$ which will imply that $\mathcal{T}\mathbf{u} : [0,1] \to M$.

Let us therefore consider $\mathcal{B}(X, M)$ as in section 2.1. By the previous remarks we have:

$$\mathcal{T} : \mathcal{B}(X, M) \to \mathcal{B}(X, M).$$

We can therefore restrict our attention to the functions $\varphi_i : M \to M$. In order for them to satisfy a Lipschitz condition on $M$, we will have

$$||\varphi_i(\vec{x}) - \varphi_i(\vec{y})|| \leq d \; ||\vec{x} - \vec{y}||, \quad \forall \vec{x}, \vec{y} \in M.$$

Since

$$||\varphi_i(\vec{x}) - \varphi_i(\vec{y})|| = ||A_i(\vec{x} - \vec{y})||,$$

it is clearly sufficient that there exists a norm $||.||$ in $\mathcal{R}^N$ such that $||A_i/_V|| < 1, i = 1, 2$, where $V$ is the hyperplane parallel to $M$ through the origin, i.e.

$$V = \{\vec{x} \in \mathcal{R}^N : \sum_{i=1}^{N} x_i = 0\}.$$

Hence, if there exists a norm $||.|| \in \mathcal{R}^N$, such that $||A_i/V|| \leq c^1 < 1, i = 1, 2$, then $\varphi_i$ are contractive with contractivity factor $c^1$ on $M$, and therefore,

$$D(\mathcal{T}\mathbf{u}, \mathcal{T}\mathbf{v}) \leq c^1 D_p(\mathbf{u}, \mathbf{v}), \; \forall \mathbf{u}, \mathbf{v} \in \mathcal{B}(X, M).$$

Hence, $\mathcal{T}$ turns out to be contractive, and there exists $\mathbf{u}^* \in \mathcal{B}(X, M)$, such that $\mathcal{T}\mathbf{u}^* = \mathbf{u}^*$.

On the other side, it is easy to check that if $\mathbf{u}$ is such that $\mathbf{u}_i(1) = \mathbf{u}_{i+1}(0)$, $1 \leq i \leq N-1$, $\mathbf{u}_1(0) = \mathbf{u}_N(1) = 0$, then $\mathcal{T}\mathbf{u}$ has the same properties. Hence, if

$$\mathcal{S} = \{\mathbf{u} \in \mathcal{B}(X, M) \; : \; \mathbf{u}_i(1) = \mathbf{u}_{i+1}(0), \quad 1 \leq i \leq N-1, \quad \mathbf{u}_1(0) = \mathbf{u}_N(1) = 0\},$$

then $\mathcal{S}$ is invariant under $\mathcal{T}$, i.e. $\mathcal{T}\mathcal{S} \subseteq \mathcal{S}$. Therefore the fixed point $\mathbf{u}^*$ of $\mathcal{T}$, lies in $\mathcal{S}$. In addition, by the definition of $\mathcal{T}$, if $\mathbf{u} \in \mathcal{S}$ is continuous , then $\mathcal{T}\mathbf{u}$ will be continuous, and hence the solution $\mathbf{u}^*$ will be continuous as well, since $\mathcal{T}^n\mathbf{u} \to \mathbf{u}^*$ in the uniform metric.

Using the correspondence (3.6), we get the following

**Proposition 3.1.1** *If there exists a norm $||.|| \in \mathcal{R}^N$, such that $||A_i/V|| < 1, i = 1, 2$, then there exists a continuous scaling function $\phi$ solution of the dilation equation*

$$\phi(x) = \sum_{k=0}^{N} c_k \phi(2x - k)$$

*obtained as the function associated to the fixed point of the operator* $\mathcal{T}$ : $\mathcal{S} \to \mathcal{S}$. *The solution can be obtained from any starting function in* $\mathcal{S}$ *by iteration of the operator* $\mathcal{T}$.

We want to remark that the existence of a solution of the dilation equation for the compactly supported case was obtained before by Daubechies (see [DL91]) but we included this proposition to show that the scaling function can be seen as the "fixed point" of the operator $\mathcal{T}$.

## 3.2 Some remarks about the dilation equation

As we showed in the preceding section, in order to find a solution to our dilation equation, it is enough for the operator $\mathcal{T}$ to have a fixed point on the subspace $\mathcal{S}$. The condition which we found is exactly the same than the one given by Daubechies [DL91]. There they used the notion of *joint spectral radius* of two matrices. This notion was introduced by Rota and Strang in [RS60]. They proved that the joint spectral radius of 2 matrices is less then 1, if and only if there exists a norm, such that both matrices are contractive in that norm. Hence our condition is equivalent to requiring that the joint spectral radius of $A_1$ and $A_2$ is less than 1 on the subspace $V$ which is the condition given by Daubechies.

There are many papers dealing with that question (see for example Heil [Hei92] for a complete list of references), in particular, Berger and Wang [BW] showed another equivalence for the joint spectral radius, which allows a somewhat easier calculation.

Our aim in this particular case is to present a solution to the dilation equation as a consequence of the previously introduced model.

This model can be applied to higher dimensional dilation equations carefully redefining the maps and subspaces. In particular, it generalizes the results of Gröchenig and Madych in [GM92] for the case that not all the coefficients $c_k$ of the two-dimensional dilation equation are equal.

The application of this model to matrix dilation equations is explored in a joint work with C. Heil [CCM94].

# 4 Application to image reconstruction

In this section we want to outline how the previous results can be used to attempt a solution to the inverse problem of fractals and other sets. The details of this application will be developed in a forthcoming paper [CM94].

The idea is the following: given a target function - signal, image, etc. we want to find two sets of maps $\{w_i\}_{1\leq i\leq r}$ and $\{\varphi_i\}_{1\leq i\leq r}$ and a suitable contractive operator, such that the fixed point of that operator is "close" to the target. Let us be more precise:

Let $\overline{u}: X \to [0,1]$ be a function defined on $X$, where $X$ will be an appropriate space. $\overline{u}$ will represent our "target function" i.e: if we want to approximate a 1-dimensional function, or a signal, $X$ will be an interval on the line, if instead we want to approximate a 2-dimensional function, or an image, $X$ will be a subset of the plane (we can generalize this to higher dimensions).

Our task is to find an operator $\mathcal{T}$ of the type described in equation (2.4) such that its fixed point $\mathbf{u}^*$ is "close" to $\overline{u}$. In order to accomplish this, we have to find:

1. $r$ the number of maps
2. $\{w_1, \ldots, w_r\}, \quad w_i : X \to X$ one-to one.
3. $\{\varphi_1, \ldots, \varphi_r\}$,

   $\varphi_i : X \times [0,1] \to [0,1]:$ such that for each fixed $x_0$, $\varphi_i(x_0, .)$ is contractive.

For simplicity of the notation, we will describe the method in dimension 1, and we take $X = [0,1]$. We first arbitrarily fix a natural number $r$ and let $w_i$ be the affine maps that define the elementary partition of the unit interval into $r$ equal parts, i.e.:

$$w_i : [0,1] \to [0,1], \quad w_i(x) = \frac{1}{r}x + \frac{i-1}{r} \quad 1 \leq i \leq r.$$

We then want to find $\varphi_i, \quad 1 \leq i \leq r$, such that

$$\begin{aligned} \varphi_i(x, \overline{u}(w_i^{-1}(x))) &= \overline{u}(x), \qquad \forall x \in \ w_i([0,1]) = [\frac{i-1}{r}, \frac{i}{r}], \text{ i.e:} \\ (4.1) \quad \varphi_i(w_i(z), \overline{u}(z)) &= \overline{u}(w_i(z)) \qquad \forall z \in [0,1]. \end{aligned}$$

This yields $r$ equations in the unknowns $\varphi_1, \ldots, \varphi_r$. We now sample $[0,1]$ in $M$ points $z_1, \ldots, z_M$ and substitute these values into equation (4.1), obtaining $M$ points of the graph of each $\varphi_i$. We can now interpolate $\varphi_i$ using some known method. If we impose the condition that the $\varphi_i's$ should be affine, we can use the best linear approximation using for example a least square approximation algorithm.

It can be seen, that the $\varphi_i's$ so obtained, are non-expansive, and hence a slight modification of them will be contractive.

We now consider $\mathcal{T}$ as in (2.4),

$$\mathcal{T}\mathbf{u} = \sup_{1\leq i\leq r} \varphi_i(x, \mathbf{u} \circ w_i^{-1}(x)),$$

where $\mathbf{u} \circ w_i^{-1}$ is defined as in (2.5)

$$u \circ w_i^{-1}(x) = \begin{cases} \mathbf{u} \circ w_i^{-1}(x) & x \in w_i(X) \\ \mathbf{u}(0) & x \notin w_i(X). \end{cases}$$

Note that in this case the operator $\mathcal{O}$ is the supreme which is non-expansive. Hence it can be seen that we are under the hypothesis the theorems 2.1.1 and 2.2.2, and therefore there exists $\mathbf{u}^*$ such that $\mathcal{T}\mathbf{u}^* = \mathbf{u}^*$.

Furthermore, it can be shown, that the distance between $\mathbf{u}^*$ and $\overline{u}$ is proportional to the error committed by taking linear approximations. Hence $\mathbf{u}^*$ is as close to $\overline{u}$ as good as the approximation of the $\varphi_i's$ is.

We are aware that this method is a very coarse first approximation - with yet surprisingly good results. Further studies would exploit different choices for the base maps, and different methods for partitioning the interval. In addition we might consider the possibility of allowing $\varphi$ to be quadratic - or even cubic splines, which would widen the possibilities immensely. The only reason why we restricted ourselves to linear functions, is because of the computation required in the other cases.

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