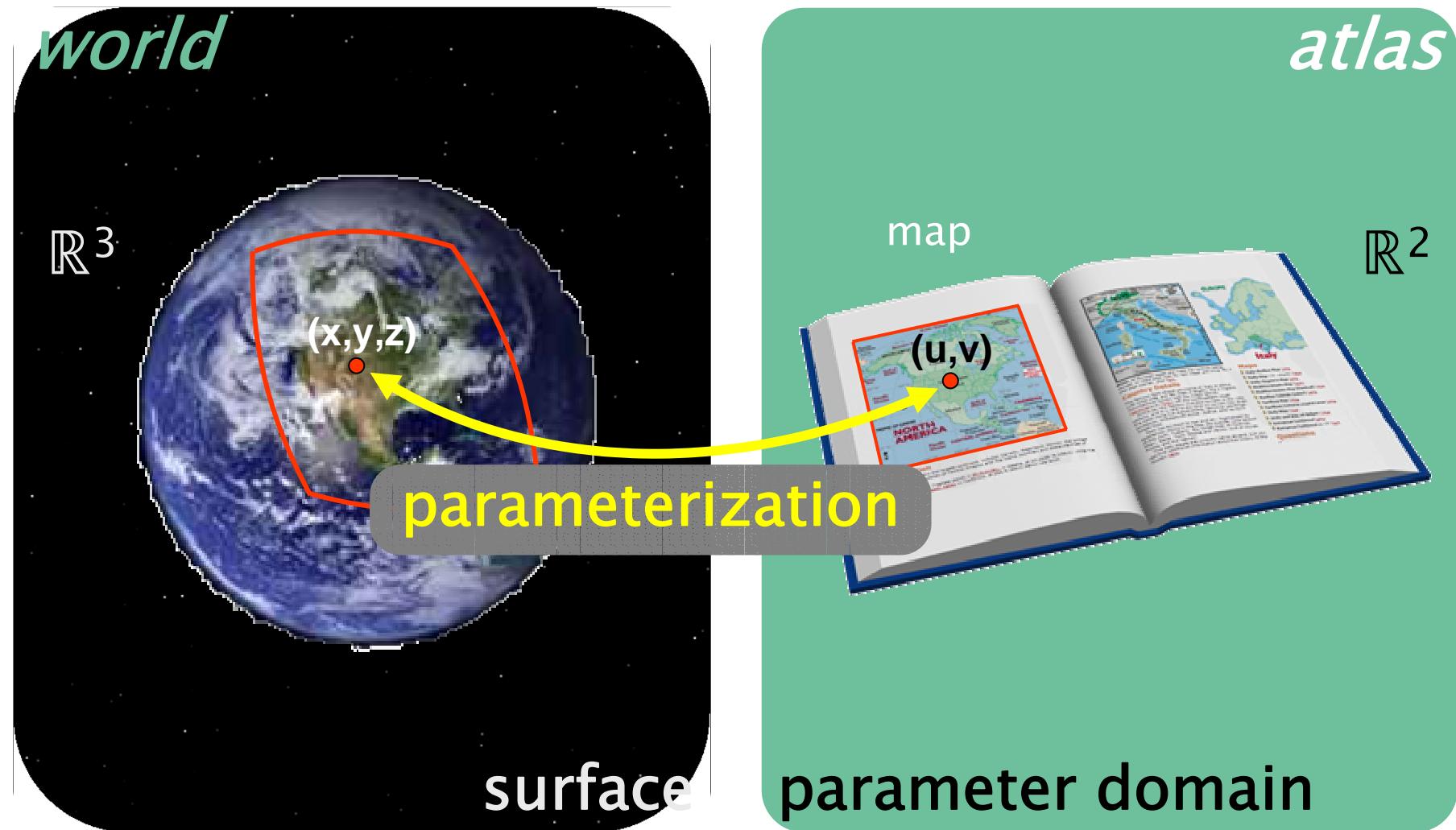


Foundations of Surface Parameterization

Kai Hormann

TU Clausthal

Introduction

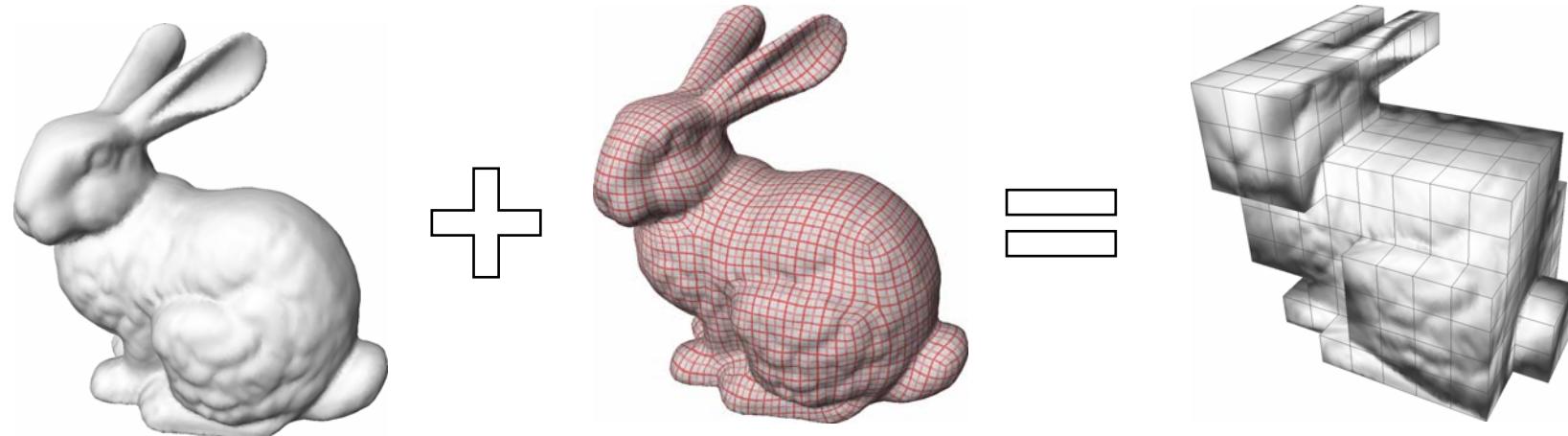


Applications

- texture mapping

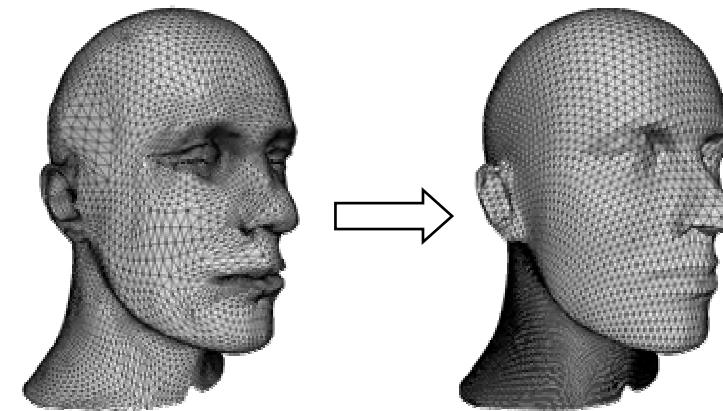
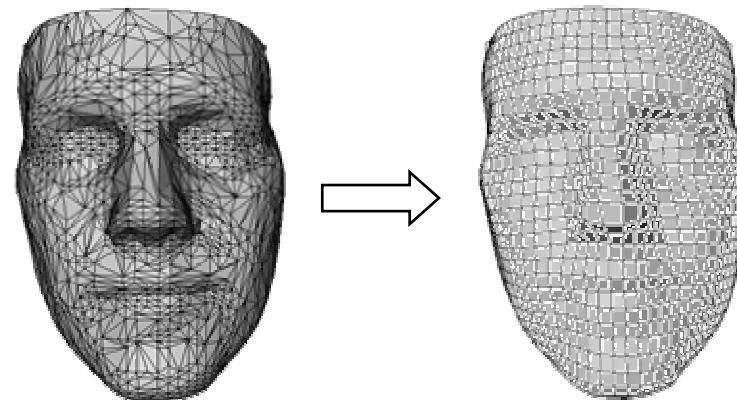


- texture synthesis

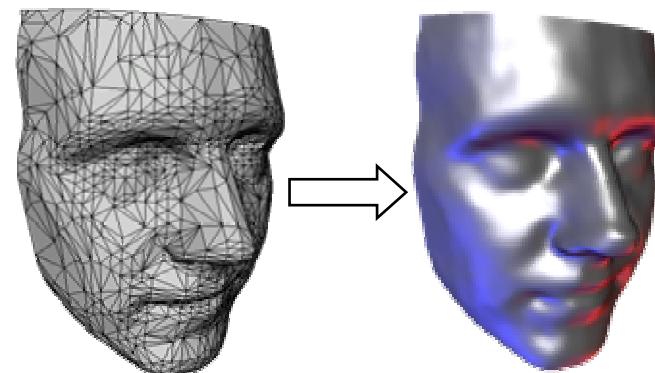
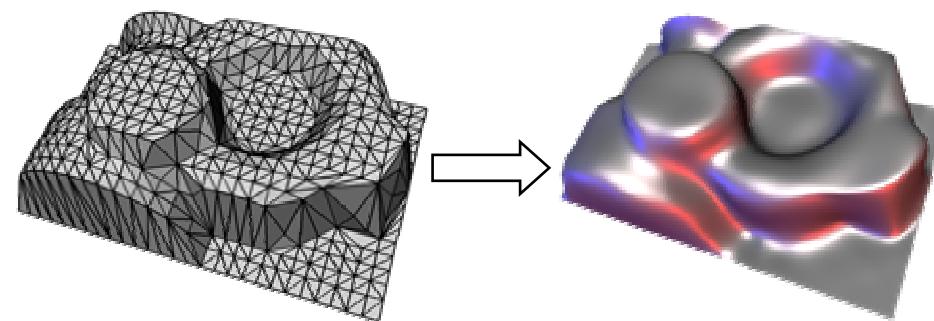


Applications

- remeshing



- surface reconstruction



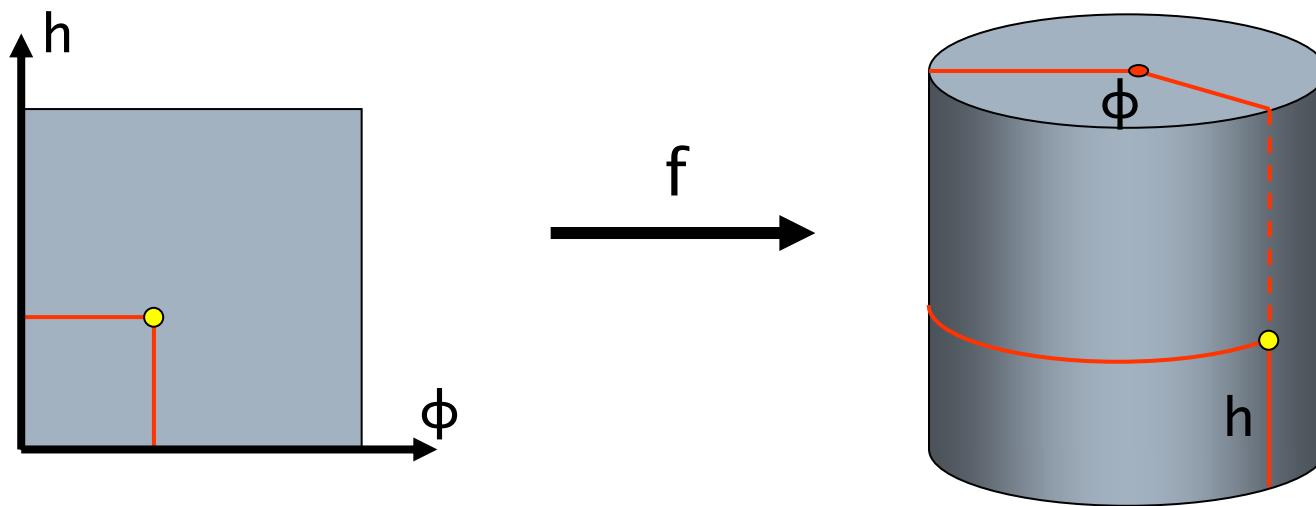
Parameterization

- **surface** $S \subset \mathbb{R}^3$
- **parameter domain** $\Omega \subset \mathbb{R}^2$
- **mapping** $f: \Omega \rightarrow S$ and $f^{-1}: S \rightarrow \Omega$



Example

cylindrical coordinates

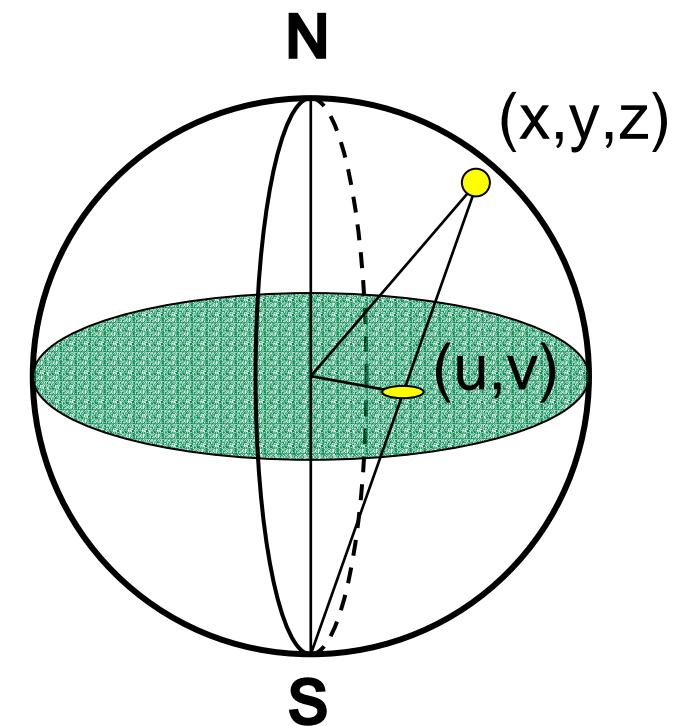


- $\Omega = \{(\phi, h) \in \mathbb{R}^2 : \phi \in [0, 2\pi), h \in [0, 1]\}$
- $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0, 1]\}$
- $f(\phi, h) = (\sin \phi, \cos \phi, h)$

Example

stereographic projection

- $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $\Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$
- $f^{-1}(x,y,z) = (x, y) / (1 + z)$
- $f(u,v) = (2u, 2v, 1 - u^2 - v^2) / (1 + u^2 + v^2)$

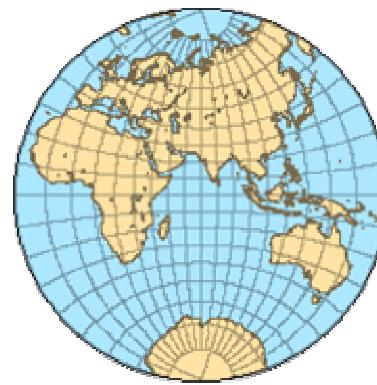


Distortion

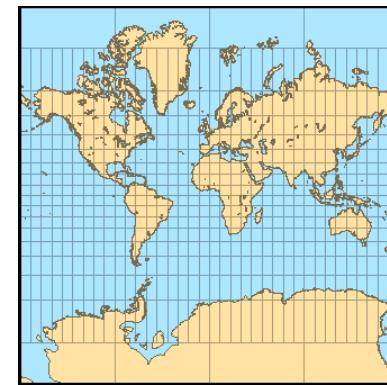
- usually, surface properties are distorted



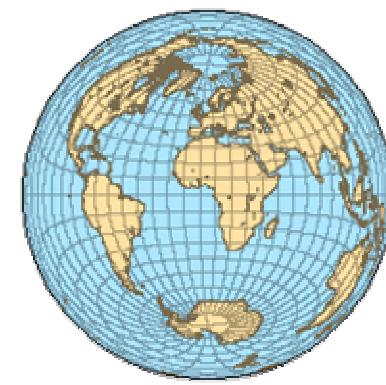
orthographic
~500 B.C.



stereographic
~150 B.C.



Mercator
1569



Lambert
1772

conformal
(angle-preserving)

equiareal
(area-preserving)

Distortion

- equiareal + conformal = **isometric**
= no distortion
- requires surface to be **developable**
 - planes
 - cones
 - cylinders

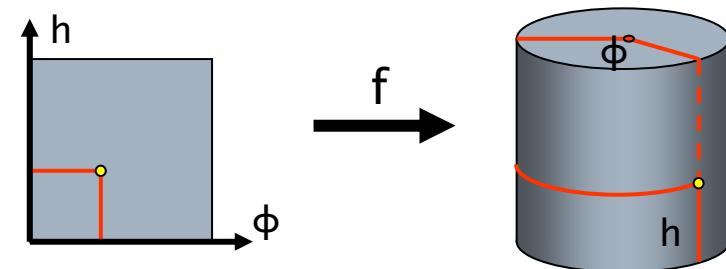


Differential Geometry Background

- **derivative** $D_f = [\partial f_i / \partial u_j]_{i,j} \in \mathbb{R}^{3 \times 2}$
- **first fundamental form** $I_f = D_f^T D_f \in \mathbb{R}^{2 \times 2}$
- **eigenvalues** λ_1, λ_2 of I_f
- **conformal** $\Leftrightarrow \lambda_1 / \lambda_2 = 1$
- **equiareal** $\Leftrightarrow \lambda_1 \cdot \lambda_2 = 1$
- **isometric** $\Leftrightarrow \lambda_1 = \lambda_2 = 1$
- note: everything defined pointwise on Ω

Example

cylindrical coordinates



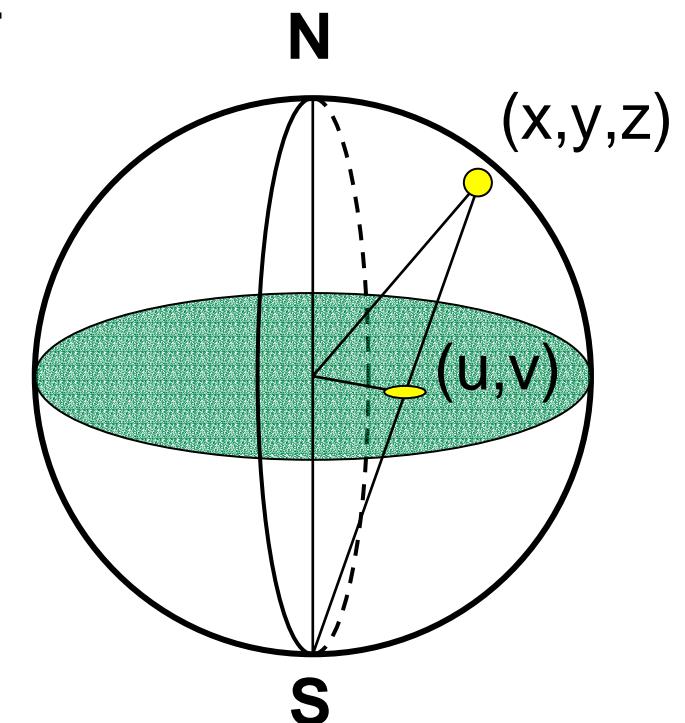
- $\Omega = \{(\phi, h) \in \mathbb{R}^2 : \phi \in [0, 2\pi), h \in [0, 1]\}$
- $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0, 1]\}$
- $f(\phi, h) = (\sin \phi, \cos \phi, h)$
- $D_f = \begin{pmatrix} \cos \phi & 0 \\ -\sin \phi & 0 \\ 0 & 1 \end{pmatrix}, \quad I_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{isometric}$

Example

stereographic projection

- $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $\Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$
- $f(u,v) = (2u, 2v, 1 - u^2 - v^2) / (1 + u^2 + v^2)$
- $\lambda_1 = \lambda_2 = 4 / (1 + u^2 + v^2)^2$

⇒ **conformal**



The Geometric Approach

- first order Taylor expansion

$$f(x) = f(y) + D_f(y) \cdot (x-y) + \dots$$

- columns of D_f span the tangent plane
- SVD (singular value decomposition)

$$D_f = U^T \cdot \Sigma \cdot V$$

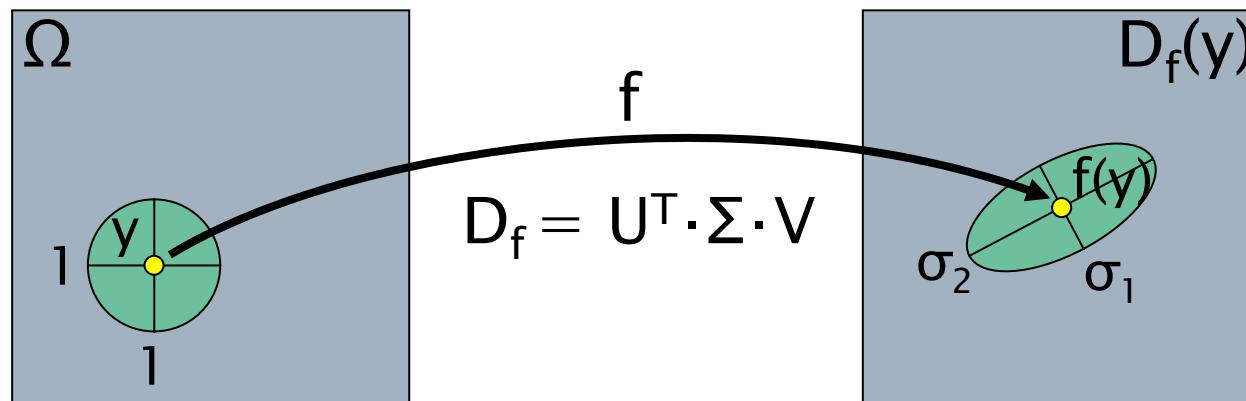
where $U \in \mathbb{R}^{3 \times 3}$, $V \in \mathbb{R}^{2 \times 2}$ orthogonal and

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} \text{ with singular values } \sigma_1 \geq \sigma_2 > 0$$



The Geometric Approach

- σ_1 and σ_2 describe local deformation in the tangent plane

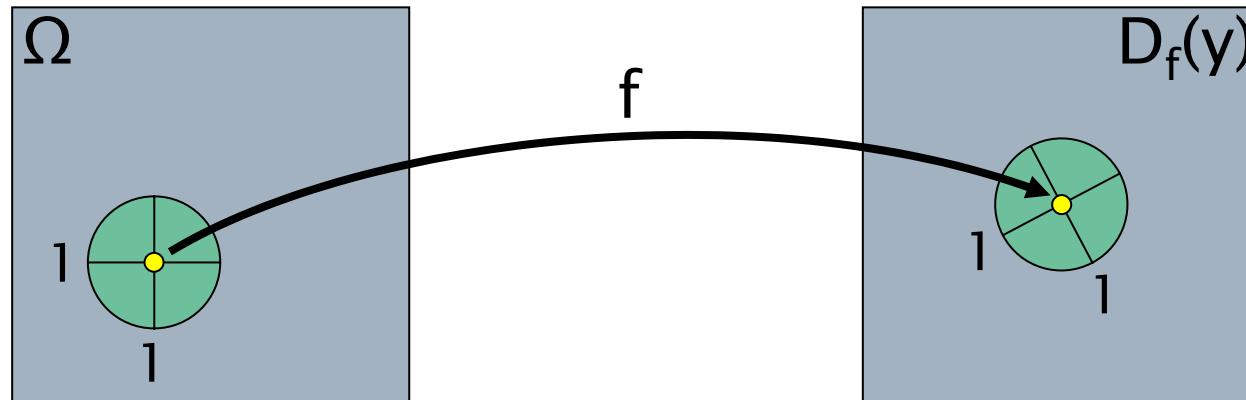


- connection to eigenvalues λ_1, λ_2 of $I_f = D_f^T D_f$

$$\lambda_i = \sigma_i^2$$

Isometric Mapping

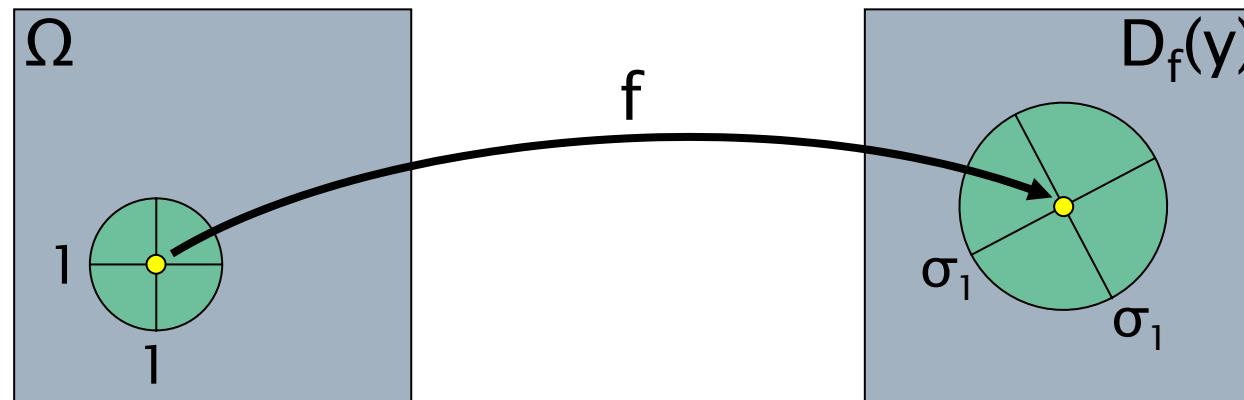
- $\sigma_1 = \sigma_2 = 1 \iff \lambda_1 = \lambda_2 = 1$



- preserves **areas, angles and lengths**

Conformal Mapping

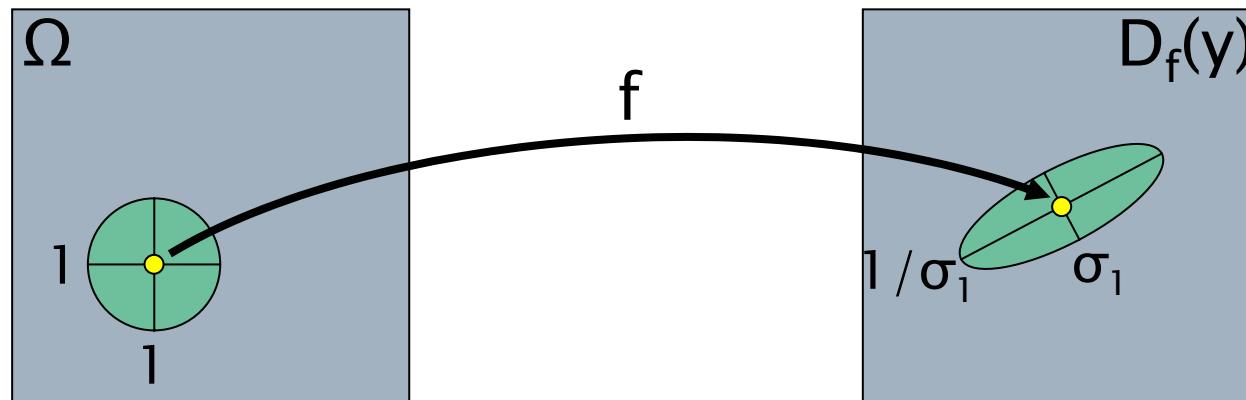
- $\sigma_1 / \sigma_2 = 1 \iff \lambda_1 / \lambda_2 = 1$



- preserves angles

Equiareal Mapping

- $\sigma_1 \cdot \sigma_2 = 1 \iff \lambda_1 \cdot \lambda_2 = 1$



- preserves **areas**

Measuring Distortion

- define a **pointwise** measure

$$E: \Omega \rightarrow \mathbb{R}$$

- typically, E depends on σ_1 and σ_2

$$E(x) = E(\sigma_1(x), \sigma_2(x))$$

- **average** over domain

$$E_f = \int_{\Omega} E(x) / \text{Area}(\Omega)$$



Examples of Pointwise Measures

- Green-Lagrange deformation tensor

$$E = \| I_f - Id \|^2_F = (\lambda_1 - 1)^2 + (\lambda_2 - 1)^2$$

minimal for isometric mappings

- conformal energy

$$E = (\sigma_1 - \sigma_2)^2 / 2$$

minimal for conformal mappings



Examples of Pointwise Measures

- MIPS energy

$$E = \sigma_1 / \sigma_2 + \sigma_2 / \sigma_1$$

minimal for conformal mappings

- energy of Degener et al. [2003]

$$E = \sigma_1 \cdot \sigma_2 + 1 / (\sigma_1 \cdot \sigma_2)$$

minimal for equiareal mappings

Examples of Pointwise Measures

- Dirichlet energy

$$E = (\sigma_1^2 + \sigma_2^2)/2$$

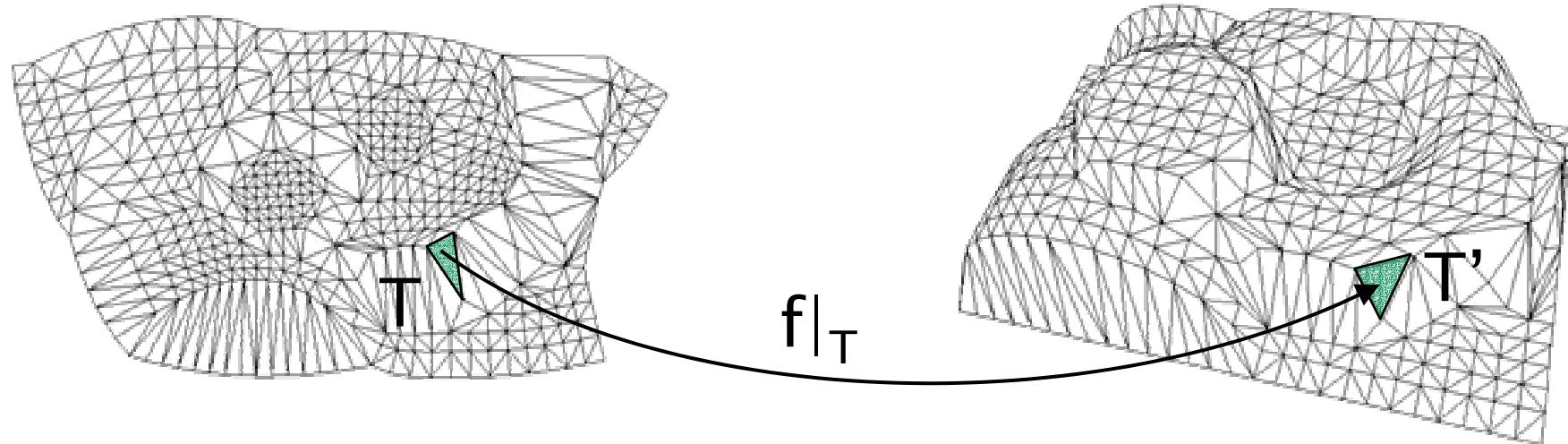
- stretch energy of Sander et al. [2001]

$$E = \sigma_1$$

- energy of Sorkine et al. [2002]

$$E = \sigma_1 + 1/\sigma_1$$

Special Case: Triangle Meshes

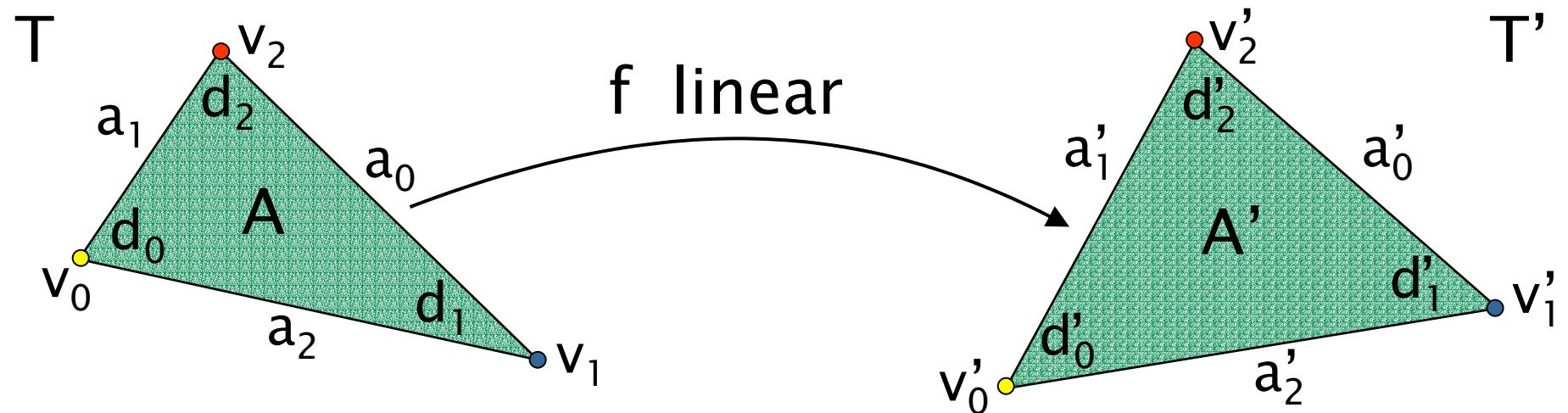


- piecewise linear **atomic maps**

$$f|_T : T \rightarrow T', x \mapsto M_T \cdot x + b_T$$

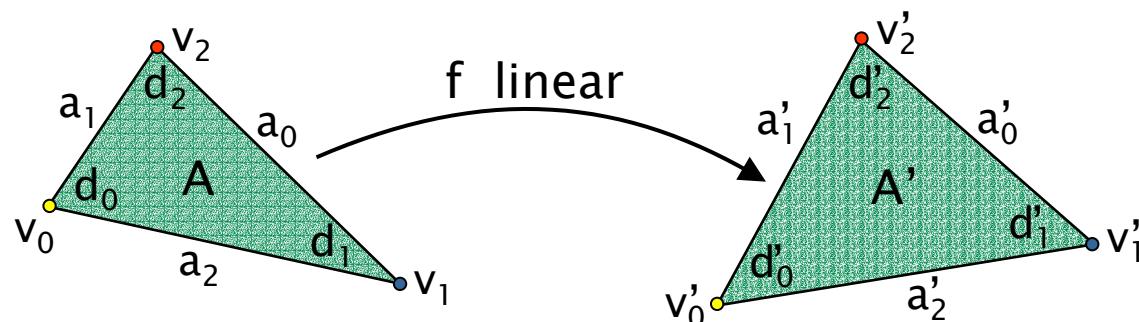
- $D_f(x) = M_T$ **constant** for $x \in T$
- how to compute **triangle distortion** $E(T)$?

Triangle Distortion



- edge lengths $a_i = \|v_{i+2} - v_{i+1}\|^2$
- dot products $d_i = \langle v_{i+1} - v_i | v_{i+2} - v_i \rangle$
- triangle area $A = \|(v_2 - v_0) \times (v_1 - v_0)\|$
- and likewise for T'

Triangle Distortion



$$a_i = \|v_{i+2} - v_{i+1}\|^2$$

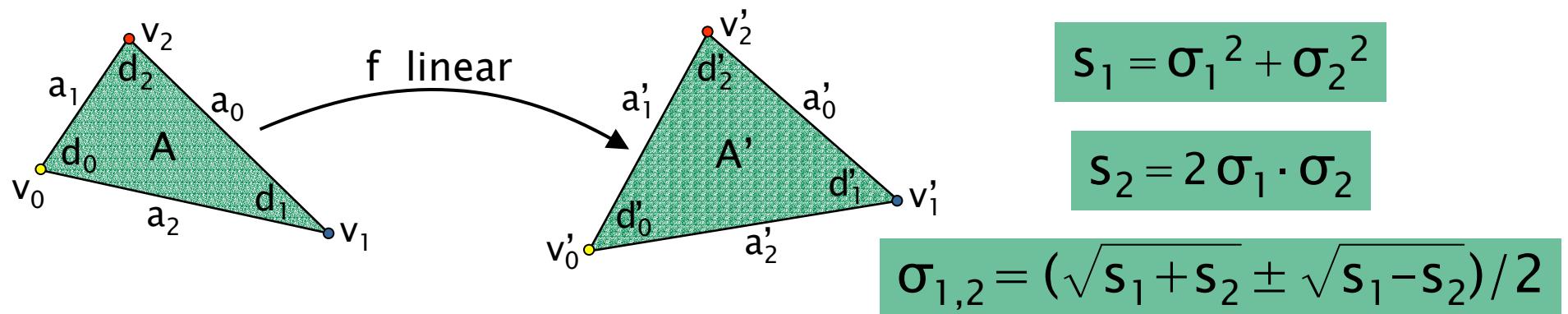
$$d_i = \langle v_{i+1} - v_i | v_{i+2} - v_i \rangle$$

$$A = \|(v_2 - v_0) \times (v_1 - v_0)\|$$

- $S_1 = \text{trace}(I_f) = \sigma_1^2 + \sigma_2^2 = (\sum_i a_i \cdot d'_i) / A^2$
- $S_2 = 2 \det(D_f) = 2 \sigma_1 \cdot \sigma_2 = 2 A' / A$

$$\sigma_{1,2} = (\sqrt{s_1 + s_2} \pm \sqrt{s_1 - s_2}) / 2$$

Computing Distortions



- formulas allow to **efficiently** compute triangle distortions
- most measures depend on s_1 and s_2 only
- for example, **conformal energy**

$$E = (\sigma_1 - \sigma_2)^2 / 2 = (s_1 - s_2) / 2$$

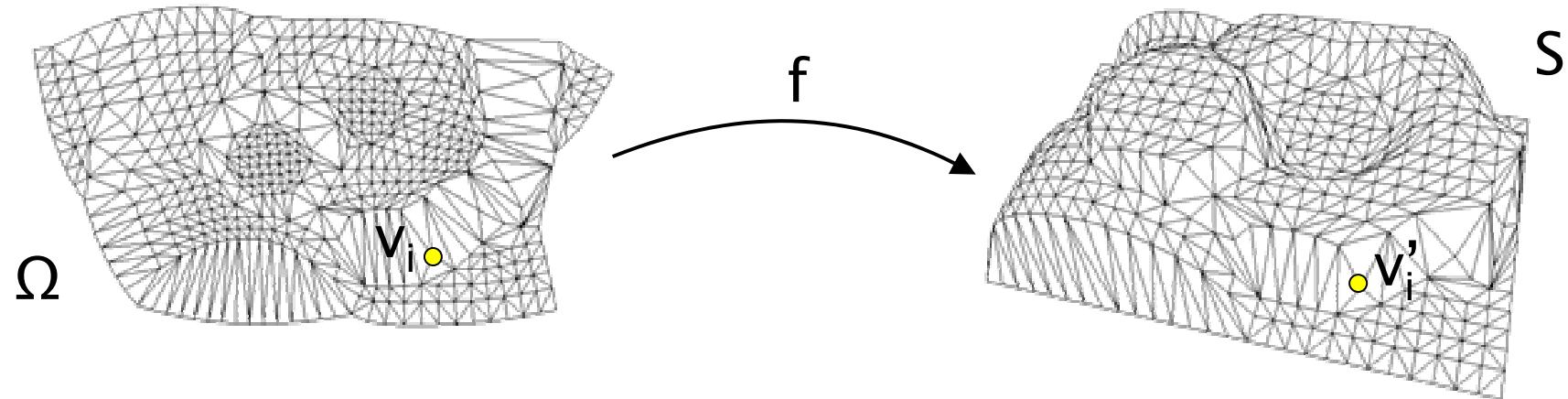
Overall Distortion

- distortion of piecewise linear parameterizations

$$\begin{aligned} E_f &= \int_{\Omega} E(x) / \text{Area}(\Omega) \\ &= \sum_T E(T) \cdot A(T) / \sum_T A(T) \end{aligned}$$



Optimizing Parameterizations



- triangle mesh S with vertices $\{v'_i\}$ given
- modify **parameter points** $\{v_i\}$ in Ω such that E_f is reduced
- solve the optimization problem

$$\min_{\{v_i\}} E_f$$

Optimizing Parameterizations

- remember $E_f = \sum_T E(T) \cdot A(T) / \sum_T A(T)$
- highly non-linear optimization problem,
because $A(T)$ depends on the $\{v_i\}$
- most methods minimize

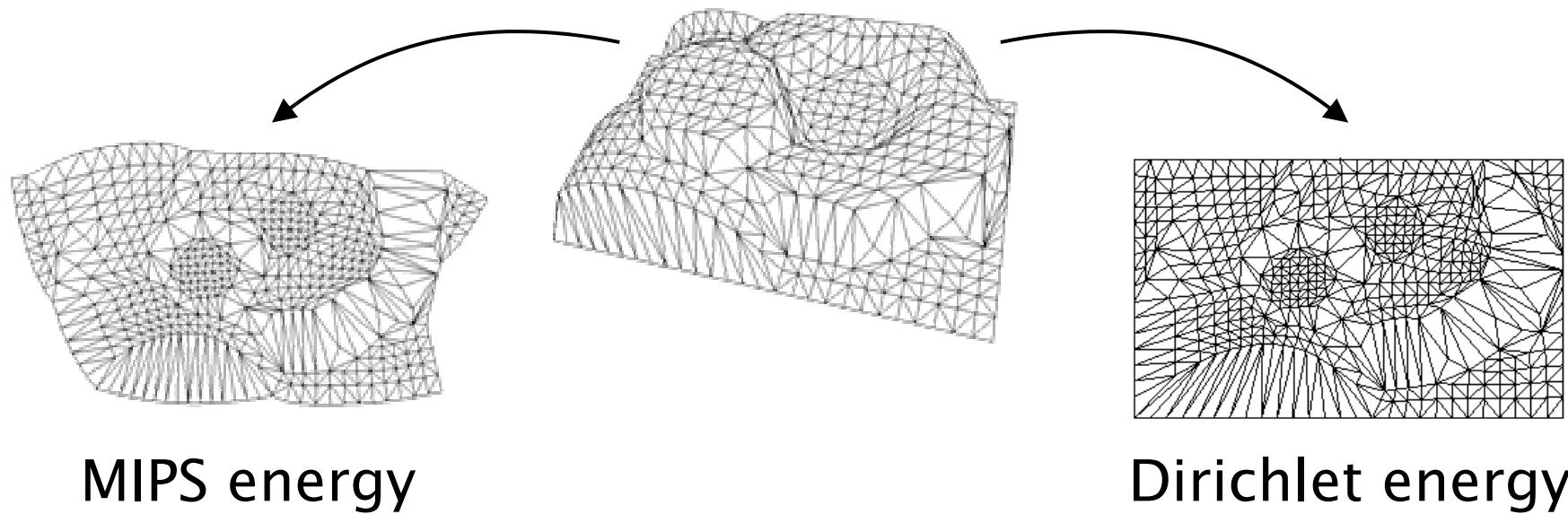
$$E_{f-1} = \sum_{T'} E(T') \cdot A(T') / \sum_{T'} A(T')$$

with constant denominator

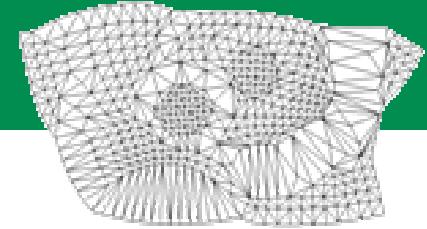


Optimizing Parameterizations

- different **pointwise measures** lead to
 - different **results**
 - different **optimization problems**

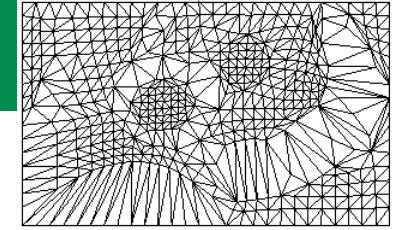


Example



- minimizing $E_{f^{-1}}$ for
 - MIPS energy $E_M = \sigma_1/\sigma_2 + \sigma_2/\sigma_1$
 - Degener's energy $E_A = \sigma_1\sigma_2 + 1/(\sigma_1\sigma_2)$
- rational quadratic in the $\{v_i\}$
- first and second derivative available
- gradient methods applicable
- initial solution required

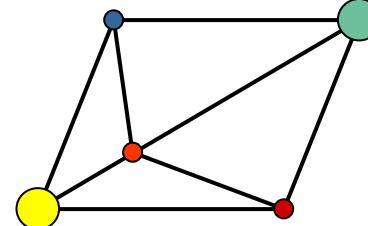
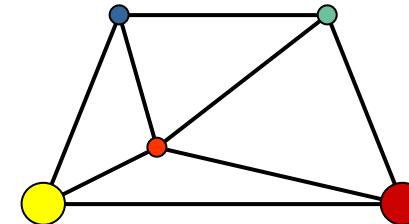
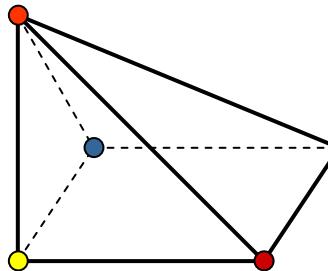
Example



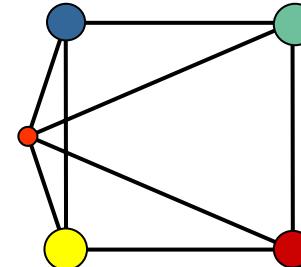
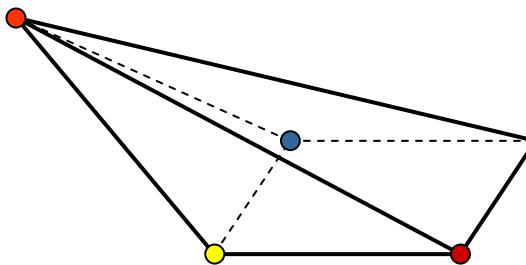
- minimizing $E_{f^{-1}}$ for
 - conformal energy $E_C = (\sigma_1 - \sigma_2)^2 / 2$
 - Dirichlet energy $E_D = (\sigma_1^2 + \sigma_2^2) / 2$
- quadratic in the $\{v_i\}$
- solving a linear system
- requires to fix some (boundary) vertices
- result depends on these fixed vertices
- result is not necessarily bijective

Example

- conformal maps



- Dirichlet maps



Which Method to Choose?

- appropriate choice depends on the specific application
- frequent problems
 - computation speed
 - find “good” tradeoff between angle and area distortion
- good heuristic $E = E_M \cdot E_A^q$, $q \approx 3$



Summary

Today we learned

- what a parameterization is
- why and how it distorts the surface
- how to measure distortion
- the simplification for triangle meshes
- how to compute a parameterization

Tomorrow we will learn

- what this is all good for

