Plausibility Measures: A General Approach for Representing Uncertainty

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Some of this work joint with Nir Friedman;
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How should we represent uncertainty?

The standard approach is probability:

- **Pros**: precise, refined tradeoffs, well-understood foundations, applications in decision theory.
How should we represent uncertainty?

The standard approach is probability:

**Pros:** precise, refined tradeoffs, well-understood foundations, applications in decision theory.

**Cons:** too precise, forces linear order, can be computationally expensive
The Alternatives . . .

Many alternatives to probability have been suggested:

- qualitative probability
- non-additive probability
- lexicographic probability system (lps)
- sets of probability measures
- (Dempster-Shafer) belief functions
- possibility theory and fuzzy logics
- $\kappa$-rankings
- . . .
**Plausibility measures** provide a single unified framework for studying representations of uncertainty. They allow us to:

- capture all previously studied approaches
- prove general results about representations of uncertainty
- get a deeper understanding of what makes a representation “good”
- provide semantics for
  - belief and belief change
  - counterfactual reasoning
  - preferential (default) reasoning
- provide a general approach to decision theory
A probability space is a tuple \((W, \mathcal{F}, \Pr)\), where

- \(W\) is a set of worlds,
- \(\mathcal{F}\) is an algebra of measurable subsets of \(W\) (a set of subsets closed under finite union and complementation), and
- \(\Pr\) is a probability measure, mapping sets in \(\mathcal{F}\) to \([0, 1]\).

A plausibility space is a direct generalization:

- Replace \(\Pr\) by a plausibility measure \(\Pl\), which maps sets in \(\mathcal{F}\) to elements of some arbitrary set \(D\) partially ordered by \(\leq\).
- Assume \(D\) contains special elements \(\bot\) and \(\top\) such that \(\bot \leq d \leq \top\), \(\Pl(\emptyset) = \bot\), and \(\Pl(W) = \top\).
Conditions on Plausibility

Pl1. $Pl(W) = \top$.

Pl2. $Pl(\emptyset) = \bot$.

Pl3. If $U \subseteq V$, then $Pl(U) \leq Pl(V)$.

These are minimal requirements for a representation of likelihood.

- Lack of structure makes everything a plausibility measure.
- Having few requirements allows us to impose additional requirements as needed.
Defaults are statements of the form $\varphi \rightarrow \psi$

- read “if $\varphi$ then typically $\psi$”.

Defaults appear often in natural language.

An (in)famous example:

- $Bird \rightarrow Fly$
- $Penguin \rightarrow \neg Fly$
- $Penguin \rightarrow Bird$

Default reasoning is not *monotonic*:

- If $A \rightarrow C$, then we may not have $A \land B \rightarrow C$

Nevertheless, it seems that it should obey certain properties.
The KLM Properties

While there is no consensus regarding the properties of defaults, there is an agreement on a common “core” of default reasoning.

Kraus, Lehmann and Magidor suggest a set of properties (the “KLM properties”) that characterize this core.

(LLE) If $\varphi \equiv \varphi'$, then from $\varphi \rightarrow \psi$ infer $\varphi' \rightarrow \psi$.

(REF) Always infer $\varphi \rightarrow \varphi$.

(RW) If $\psi \Rightarrow \psi'$, then from $\varphi \rightarrow \psi$ infer $\varphi \rightarrow \psi'$.

(AND) From $\varphi \rightarrow \psi_1$ and $\varphi \rightarrow \psi_2$ infer $\varphi \rightarrow \psi_1 \land \psi_2$.

(OR) If $\varphi_1 \rightarrow \psi$ and $\varphi_2 \rightarrow \psi$ infer $\varphi_1 \lor \varphi_2 \rightarrow \psi$.

(CM) From $\varphi \rightarrow \psi_1$ and $\varphi \rightarrow \psi_2$ infer $\varphi \land \psi_1 \rightarrow \psi_2$. 
Many (quite different) semantics for default reasoning are characterized by the KLM properties.

Plausibility structures help explain why.
A preferential structure [KLM] is a tuple

\[ P = (W, \prec, \pi) \], where \( \pi \) maps each world in \( W \) to a truth assignment, and \( \prec \) is a partial order on \( W \).

- \( x \prec y \) denotes that \( x \) is more preferred than \( y \).
- For now, we assume that \( \prec \) is well-founded.
- May have two different worlds associated with the same truth assignment \( (\pi(w) = \pi(w') \text{ for } w \neq w') \)

A preferential structure provides semantics for defaults:

- \( P \models \varphi \rightarrow \psi \) if the most plausible worlds (i.e., minimal according to \( \prec \)) that satisfy \( \varphi \) also satisfy \( \psi \).
Lehmann and Magidor define entailment w.r.t. to a class $C$ of preferential structures:

\[ \Delta \text{ entails } \varphi \rightarrow \psi \text{ w.r.t. } C \text{ if every } P \in C \text{ that satisfies all the defaults in } \Delta \text{ also satisfies } \varphi \rightarrow \psi. \]

**Theorem:** [KLM] The KLM properties provide a sound and complete axiomatization of entailment w.r.t. preferential structures.

\[ \Delta \text{ entails } \varphi \rightarrow \psi \text{ w.r.t. all preferential structures iff } \varphi \rightarrow \psi \text{ can be derived from } \Delta \text{ using the KLM properties.} \]
An alternative semantics of defaults is in terms of extreme probabilities [Adams; Pearl].

Trying to capture the intuition that $\varphi \rightarrow \psi$ means $\Pr(\psi | \varphi) \approx 1$.

But how do we capture “$\approx 1$”? 

A parametrized probability distribution (PPD) is a sequence $(\Pr_n)$ of probability distributions on some space $W$.

$$(W, (\Pr_n), \pi) \models \varphi \rightarrow \psi \text{ if } \lim_{n \rightarrow \infty} \Pr_n(\psi | \varphi) = 1$$

Intuitively, this means that (according to $(\Pr_n)$) the probability of the default is arbitrarily close to 1.

Note: $\Pr(\varphi) = \Pr(\{w \in W : w \models \varphi\})$. 
\( \Delta \epsilon\text{-entails} \) a default \( \varphi \rightarrow \psi \) if

- for every family \( (\mathcal{P}_n) \), if \( (\mathcal{P}_n) \models \Delta \) then \( (\mathcal{P}_n) \models \varphi \rightarrow \psi \).
- as the confidence in \( \Delta \) goes to 1, so does the confidence in \( \varphi \rightarrow \psi \).

**Theorem:** [Geffner] The KLM properties provide a sound and complete axiomatization of entailment w.r.t. \( \epsilon\)-entailment.

It is remarkable that two totally different interpretations of defaults yield identical sets of conclusions and identical sets of reasoning machinery.

– J. Pearl, 1989
**Other Semantics for Defaults**

*Possibility measures* [Zadeh; Dubois/Prade]. These are functions that assign to each event a degree of *possibility* between 0 and 1.

\[ P(\emptyset) = 0, \ P(W) = 1, \ P(A \cup B) = \max(P(A), P(B)) \]

\((W, P, \pi) \models \varphi \rightarrow \psi \text{ if } P(\varphi) = 0 \text{ or } P(\varphi \land \psi) > P(\varphi \land \neg \psi)\).

*κ-rankings* [Spohn, Goldszmidt/Pearl]. These are functions that assign to each event an ordinal that measures degree of *surprise*.

\[ \kappa(W) = 0, \ \kappa(\emptyset) = \infty, \ \kappa(A \cup B) = \min(\kappa(A), \kappa(B)) \]

\((W, \kappa, \pi) \models \varphi \rightarrow \psi \text{ if } \kappa(\varphi) = \infty \text{ or } \kappa(\varphi \land \psi) < \kappa(\varphi \land \neg \psi)\).

The KLM properties again characterize entailment for possibility measures and κ-rankings.
Why do we always get the KLM properties?

A plausibility structure is a tuple $PL = (W, Pl, \pi)$, where $(W, Pl)$ is a plausibility space.

$PL \models \varphi \rightarrow \psi$ if $Pl(\varphi) = \bot$ or $Pl(\varphi \land \psi) > Pl(\varphi \land \neg \psi)$.

This is just the obvious generalization of the definition used for possibility and $\kappa$-ranking.

if $Pl$ is a probability function $Pr$, then

$(W, Pr, \pi) \models \varphi \rightarrow \psi$ iff $Pr(\varphi) = 0$ or $Pr(\psi|\varphi) > 1/2$.

Key observation: We can map preferential structures and PPDs into plausibility structures while still preserving the semantics of defaults.
Given a partial order $\prec$ on $W$, we define a plausibility measure $\text{Pl}_\prec$ on $W$ with the following properties:

1. If $x \prec y$, then $\text{Pl}_\prec(\{y\}) < \text{Pl}_\prec(\{x\})$.

2. If $\text{Pl}_\prec(\{a\}) \leq \text{Pl}_\prec(B)$ for all $a \in A$, then $\text{Pl}_\prec(A) \leq \text{Pl}_\prec(B)$.

3. $\text{Pl}_\prec$ is the least plausibility measure satisfying 1 and 2.

We can think of $\text{Pl}_\prec$ as extending $\prec$ so that it is an ordering on sets, not just on elements.

**Theorem:** $(W, \prec, \pi) \models \varphi \rightarrow \psi$ iff $(W, \text{Pl}_\prec, \pi) \models \varphi \rightarrow \psi$. 
From $\epsilon$-Semantics to Plausibility

Given a PPD $PR = (Pr_1, Pr_2, \ldots)$ defined on $W$, we can define a plausibility measure $Pl_{PR}$ on $W$ such that

$$Pl_{PR}(A) \leq Pl_{PR}(B) \text{ iff } \lim_{n \to \infty} Pr_n(B|A \cup B) = 1.$$ 

Intuitively, $Pl_{PR}(A) \leq Pl_{PR}(B)$ iff $B$ is “almost all” of $A \cup B$.

**Theorem:** $(W, PR, \pi) \models \varphi \rightarrow \psi$ iff $(W, Pl_{PR}, \pi) \models \varphi \rightarrow \psi$. 
It is easy to see that every plausibility structure satisfies LLE, RW, and REF. In general, they do not satisfy AND, OR, or CM.

**Example:** It is easy to construct a probability function such that
\[ \Pr(q_1|p) > \frac{1}{2}, \Pr(q_2|p) > \frac{1}{2}, \text{ and } \Pr(q_1 \land q_2|p) < \frac{1}{2}. \]

What does it take to make a plausibility measure satisfy the KLM properties?
Getting the AND Rule

To get the AND rule, reverse engineering shows we need

\textbf{A2′.} For all sets \(A\), \(B_1\), and \(B_2\), if

\[ \Pr(A \cap B_1) > \Pr(A \cap \overline{B_1}) \quad \text{and} \]
\[ \Pr(A \cap B_2) > \Pr(A \cap \overline{B_2}) \]
\[ \text{then} \quad \Pr(A \cap B_1 \cap B_2) > \Pr(A \cap (\overline{B_1 \cap B_2})). \]

**Proposition:** \(\Pr\) satisfies \(A2′\) iff \(\Pr\) satisfies

\textbf{A2.} For all pairwise disjoint sets \(A\), \(B\), and \(C\), if

\[ \Pr(A \cup B) > \Pr(C) \quad \text{and} \]
\[ \Pr(A \cup C) > \Pr(B), \]
\[ \text{then} \quad \Pr(A) > \Pr(B \cup C). \]
We could reverse engineer rules for CM and OR, but we don’t need to.

- **CM** follows from A1 and A2
- **OR** almost does too

If \( \text{Pl} \) satisfies A1 and A2, \( \text{Pl} \models \varphi_1 \rightarrow \psi \), \( \text{Pl} \models \varphi_2 \), and either \( \text{Pl}(\varphi_1) \neq \bot \) or \( \text{Pl}(\varphi_2) \neq \bot \), then \( \text{Pl} \models (\varphi_1 \lor \varphi_2) \rightarrow \psi \).

To get the full OR rule, we need

**A3.** If \( \text{Pl}(A) = \text{Pl}(B) = \bot \), then \( \text{Pl}(A \cup B) = \bot \).

A plausibility structure \((W, \text{Pl}, \pi)\) is **qualitative** if \( \text{Pl} \) satisfies A2 and A3. Let \( \mathcal{P}^{QPL} \) consist of all qualitative plausibility structures.

**Theorem:** If \( \mathcal{P} \subseteq \mathcal{P}^{QPL} \), then the KLM properties are sound in \( \mathcal{P} \).
Why we always get KLM

**Theorem:** The following sets of plausibility are all qualitative:

- \( \mathcal{P}^\kappa : \kappa \) rankings
- \( \mathcal{P}^{Poss} \): possibility measures
- \( \mathcal{P}^P \): preferential structures (using the mapping)
- \( \mathcal{P}^r \): modular preferential structures
- \( \mathcal{P}^\epsilon \): PPDs (using the mapping)

**Proof:** E.g., to see that possibility measures satisfy A2, suppose

- \( A, B, C \) disjoint; \( P(A \cup B) > P(C) \); \( P(A \cup C) > P(B) \).

Then elements of maximum possibility in \( A \cup B \cup C \) must all be in \( A \).

**Conclusion:** \( P(A) > P(B \cup C) \).
As long as $\mathcal{P} \subseteq \mathcal{P}^{QPL}$, the KLM properties are sound in $\mathcal{P}$. If we take “too small” a subset of $\mathcal{P}^{QPL}$, we may get extra properties.

**Def:** A set $\mathcal{P}$ is *rich* if for every sequence of mutually exclusive formulas $\varphi_1, \ldots, \varphi_n$, there is a plausibility structure $(W, \mathrm{Pl}, \pi) \in \mathcal{P}$ such that

$$\mathrm{Pl}(\varphi_1) > \cdots > \mathrm{Pl}(\varphi_n) = \mathrm{Pl}(\emptyset).$$

**Theorem:** A set $\mathcal{P}$ of qualitative plausibility structures is rich iff the KLM properties are sound and complete w.r.t. $\mathcal{P}$.

**Theorem:** Each of $\mathcal{P}^{Poss}$, $\mathcal{P}^{\kappa}$, $\mathcal{P}^{\epsilon}$, $\mathcal{P}^{p}$, and $\mathcal{P}^{r}$ is rich.

**Bottom line:** $A2, A3 +$ richness is what we need for the KLM properties to characterize default reasoning.
Counterfactual Reasoning

Counterfactual reasoning is used all the time in day-to-day reasoning:

- If he had only stopped at the stop sign, he would not have had the accident.
- If Oswald had not shot Kennedy, Kennedy would still be alive today.

It is also critical in analyzing causality:

- Even if it weren’t raining and I wasn’t drunk, I wouldn’t have had the accident if the brakes weren’t faulty.
- Therefore the faulty brakes caused the accident.
Semantics for Counterfactuals

Read $\varphi \succ \psi$ as “if $\varphi$ were true, then $\psi$ would be true”.

- Idea (Lewis, Stalnaker): $\varphi \succ \psi$ is true at a world $w$ if, at the closest worlds to $w$ where $\varphi$ is true, $\psi$ is also true.

Alternate idea: $\varphi \succ \psi$ is true at $w$ if $\varphi$ is highly plausible given $\psi$, according to the plausibility measure used at $w$.

- Now need a possibly different plausibility measure $\text{Pl}_w$ for each world $w$.

But what properties should $\text{Pl}_w$ have?

- That depends on the desired properties of counterfactuals.
Properties of Counterfactuals

C0. \((\varphi \land \varphi \succ \psi) \implies \psi\).

C1. \(\varphi \succ \varphi\).

C2. \(((\varphi \succ \psi_1) \land (\varphi \succ \psi_2)) \implies (\varphi \succ (\psi_1 \land \psi_2))\).

C3. \(((\varphi_1 \succ \psi) \land (\varphi_2 \succ \psi)) \implies ((\varphi_1 \lor \varphi_2) \succ \psi)\).

C4. \(((\varphi_1 \succ \varphi_2) \land (\varphi_1 \succ \psi)) \implies ((\varphi_1 \land \varphi_2) \succ \psi)\).

C5. If \(\psi_1 \implies \psi_2\), then \((\varphi \succ \psi_1) \implies (\varphi \succ \psi_2)\).

C6. If \(\varphi_1 \iff \varphi_2\) then \(\varphi_1 \succ \psi \iff \varphi_2 \succ \psi\).

C1–6 correspond to REF, AND, OR, and CM.

To capture them, we need to assume that \(P_{lw}\) is qualitative (satisfies A2 and A3).
C0. \((\varphi \land \varphi \succ \psi) \Rightarrow \psi\).

C0 essentially says that the current world is the most plausible according to \(\text{Pl}_w\).

- \(w\) is the “closest” world to itself

This is easy to formalize:

- If \(w \in A\) and \(w \notin B\), then \(\text{Pl}_w(A) > \text{Pl}_w(B)\).

**Bottom line:** We can capture counterfactual reasoning using plausibility measures:

- This leads to different intuitions regarding counterfactuals.
- It also leads to simpler proofs of completeness.
Belief Change

How should we revise our beliefs in the light of unexpected information.

- Probabilistically, this amounts to conditioning on an event of measure 0

Alchourrón, Gärdenfors, and Makinson initiated a study of this topic, proposing various properties that revision should satisfy.

- Since then, many approaches to belief revision have been proposed

Again, all can be understood in terms of conditioning plausibility measures.
Decision Theory

The standard rule for decision making is maximizing expected utility (meu). But there are many others.

- minimax
  - Choose the action that gives the best worst-case payoff
- regret minimization
- ...

Plausibility provides a general framework for defining decision rules.
There is a set $A$ of possible actions

For each action $a \in A$, there is an associated utility:

$u_a(w)$: the utility of performing action $a$ in world $w$
Assume that there are two partially ordered sets

- $D$ – the set of plausibility values
- $D'$ – the set of utility values

$\oplus : D' \times D' \rightarrow D'$
$\otimes : D' \times D \rightarrow D'$

Given plausibility measure $\text{Pl}$ with range $D$, define expectation the obvious way:

$$E_{\text{Pl}}(X) = \oplus_{x \in X} x \otimes \text{Pl}(X = x)$$

Using this definition, can define expected utility maximization with respect to a plausibility measure in the obvious way.
Generalized expected utility

Every decision rule can be viewed as an instance of maximising expected utility (with respect to an appropriate plausibility measure and utility):

**Theorem:** Given an arbitrary preference order \( > \) on actions, there is a plausibility measure \( \mathcal{P} \), utility \( u \), \( \oplus \), and \( \otimes \) such that \( a > a' \) iff

\[
E_{\mathcal{P}}(u_a) > E_{\mathcal{P}}(u_{a'})
\]

**Bottom line:** whatever the decision rule, we can always think in terms of maximising expected (plausibilistic) utility
Example: capturing minimax

- Take $D = \{0, 1\}$ and $D' = IR$.
- Let $P_{mm}(U) = 1$ if $U \neq \emptyset$; $P_{mm}(\emptyset) = 0$.
- Let $\oplus$ be min and let $\otimes = \times$.

$E_{P_{mm}}(u_a)$ is the utility of worst-case outcome when performing $a$

- minimax = choosing the action $a$ that maximizes $E_{P_{mm}}(u_a)$. $a$ has higher worst-case utility than $a'$ iff $E_{P_{mm}}(u_a) > E_{P_{mm}}(u_{a'})$. 
Plausibility provides a general framework for developing a theory of representing uncertainty:

- Allows us to understand what is needed to capture (conditional) belief, counterfactual reasoning, default reasoning, belief change, and decision making.

- Can also examine conditioning and independence in a very general setting.

- Key step to providing compact representations of plausibility measures using Bayesian networks

- By using plausibility, we can “engineer” an appropriate representation of uncertainty.